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Quantum multi-particle interference due to a single-photon state

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Abstract. Quantum interference of a multi-particle is demonstrated in a simple arrangement in which the photon distribution at the output of a lossless beamsplitter for an N-photon state input at one of its input ports is dramatically changed by a single-photon state input at the other port. By a simple argument, we find that the effect persists even if the N-photon state is replaced by an arbitrary state of light.

The interference effect of light has played an important role in the conceptual development of quantum theory. Richard Feynman once wrote [1] that the Young's double-slit experiment 'has in it the heart of quantum mechanics'. But the phenomena of light interference do not simply stop at Young's double-slit experiment and its variations. Much richer phenomena, especially quantum phenomena, occur in higher-order interference [2–7] when there is more than one particle involved in the process. For example, Greenberger *et al* [8] recently proposed a new demonstration of locality violation by quantum theory with the superposition state of three or more particles.

What motivates this work is the fact that homodyne detection of a single-photon state is completely different from that of a vacuum state, which was pointed out by Yurke and Stoler [9] and confirmed by Vogel and Grabow [10]. Since homodyne detection involves a strong classical coherent field, it is surprising that the existence of only a single photon can totally change an outcome that is supposed to be dominated by the classical field. Therefore, one may guess that such a dramatic non-classical effect must be the manifestation of some fundamental principles of quantum mechanics. In this paper, we will show that it is the quantum multi-particle interference effect that gives rise to the dramatic result in the homodyne detection of a single-photon state. Furthermore, we will show that the interference effect persists even if the coherent field is replaced by a field in arbitrary state.

We start by considering the interference between a single-photon state and an N-photon state via a beamsplitter (figure 1). But let us first look at the situation without the single-photon state. It is well known that when a number of particles, say N, enter a 50 : 50 lossless beamsplitter from one input port, each particle is randomly sent to the two output ports with equal chance (50%), resulting in the simple Bernoulli binomial probability distribution as

$$P_0(N_1, N_2) = \frac{N!}{2^N N_1! N_2!} \delta_{N_1+N_2,N} \tag{1}$$

where N_1 is the number of particles exiting from port 1 while N_2 is for port 2. In the case of photons, the above result suggests that photons act independently as classical particles.

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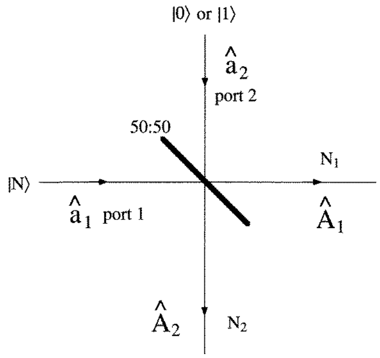

Figure 1. Interference between an N-photon and a single-photon state via a beamsplitter.

What will happen if we then let a single-photon state enter the other input port (port 2) of the beamsplitter (as seen in figure 1)? The outcome from classical particle theory is not much different from equation (1). Because the single photon is independent of the other N photons when it acts as a classical particle, we simply add the probabilities to obtain

$$P_1^{\mathrm{cl}}(N_1, N_2) = \frac{1}{2}\frac{N!}{2^N(N_1-1)!N_2!} + \frac{1}{2}\frac{N!}{2^N(N_2-1)!N_1!} = \frac{(N+1)!}{2^{N+1}N_1!N_2!} \tag{2}$$

which is in exactly the same form as in (1). Therefore the existence of the single photon at the other port does not influence the photon probability distribution at all. The single photon from port 2 acts as if it were part of the $N+1$ photons from port 1.

On the other hand, the outcome is totally different if we treat photons as quantum particles. To demonstrate the principle of quantum interference, let us consider the case when N is an odd integer and $N_1 = N_2 = (N+1)/2$. The principle of quantum mechanics requires us to add not the probabilities but the probability amplitudes, which have two contributions here: (i) the single photon input at port 2 goes to output port 1 while $N_1 - 1 = (N-1)/2$ from the N photons go to output port 1 and $N_2 = (N+1)/2$ photons to port 2, or (ii) the single photon goes to output port 2 while $N_1 = (N+1)/2$ photons go to output port 1 and $N_2 - 1 = (N-1)/2$ photons to port 2. From equation (1), we find that these two possibilities have equal probability, thus their probability amplitudes have equal absolute value. For their phases, however, because there is a $\pi/2$ phase shift for the reflected field and no phase shift for the transmitted one at a symmetric beamsplitter, the total phase shift for the $N+1$ photons at the output ports will be different for the two possibilities. Referring to figure 1, we find that the total phase shift for the first possibility is $\varphi_1 = \pi/2 + N_2\pi/2 = (N+3)\pi/4$ while for the second possibility, $\varphi_2 = (N_2-1)\pi/2 = (N-1)\pi/4$. Thus $\varphi_1 - \varphi_2 = \pi$ and the two probability amplitudes will cancel each other, resulting in zero probability for $N_1 = N_2 = (N+1)/2$. This result is completely different from that of classical particle theory in equation (2). As seen above, the probability cancellation at $N_1 = N_2$ results from the quantum interference of $N+1$ particles. A special case of $N = 1$ was demonstrated in two-photon interference [11].

For the other cases when $N_1 \neq N_2$, a similar quantum interference effect persists, but because the probabilities for the two cases are not equal, they do not completely cancel each other. We can find the probability distribution $P_1(N_1, N_2)$ by using the formula [12]

$$P_1(N_1, N_2) = \left\langle : \frac{(\hat{A}_1^\dagger \hat{A}_1)^{N_1}}{N_1!} \mathrm{e}^{-\hat{A}_1^\dagger \hat{A}_1} \frac{(\hat{A}_2^\dagger \hat{A}_2)^{N_2}}{N_2!} \mathrm{e}^{-\hat{A}_2^\dagger \hat{A}_2} : \right\rangle \tag{3}$$

where

$$\hat{A}_1 = (\hat{a}_1 + \mathrm{i}\hat{a}_2)/\sqrt{2} \qquad \hat{A}_2 = (\hat{a}_2 + \mathrm{i}\hat{a}_1)/\sqrt{2}$$

are the annihilation operators for the output modes for a symmetric lossless beamsplitter. The input modes represented by $\hat{a}_1$, $\hat{a}_2$ are in the state $|\Phi\rangle = |N\rangle_1|1\rangle_2$. After some lengthy calculation (see the appendix), we have

$$P_1(N_1, N_2) = \frac{N!(N_1 - N_2)^2}{2^{N+1}N_1!N_2!}\delta_{N_1+N_2,N+1}\,. \tag{4}$$

Equation (4) can also be derived from the general formula given by Campos *et al* in equations (48) of [13] for arbitrary numbers $\{n_1, n_2\}$ of input photons, with the setting of $\tau = \frac{1}{2}, n_1 = N, n_2 = 1$. Notice that when N is an odd integer, $P_1(N_1, N_2) = 0$ for $N_1 = N_2 = (N + 1)/2$, exactly the same as predicted from the simple argument in the previous paragraph. Actually, quantum probability cancellation makes the whole probability distribution in (4) different from that in (1), as seen in figure 2. The quantum interference plays a crucial role here in the difference between quantum mechanics and classical mechanics.

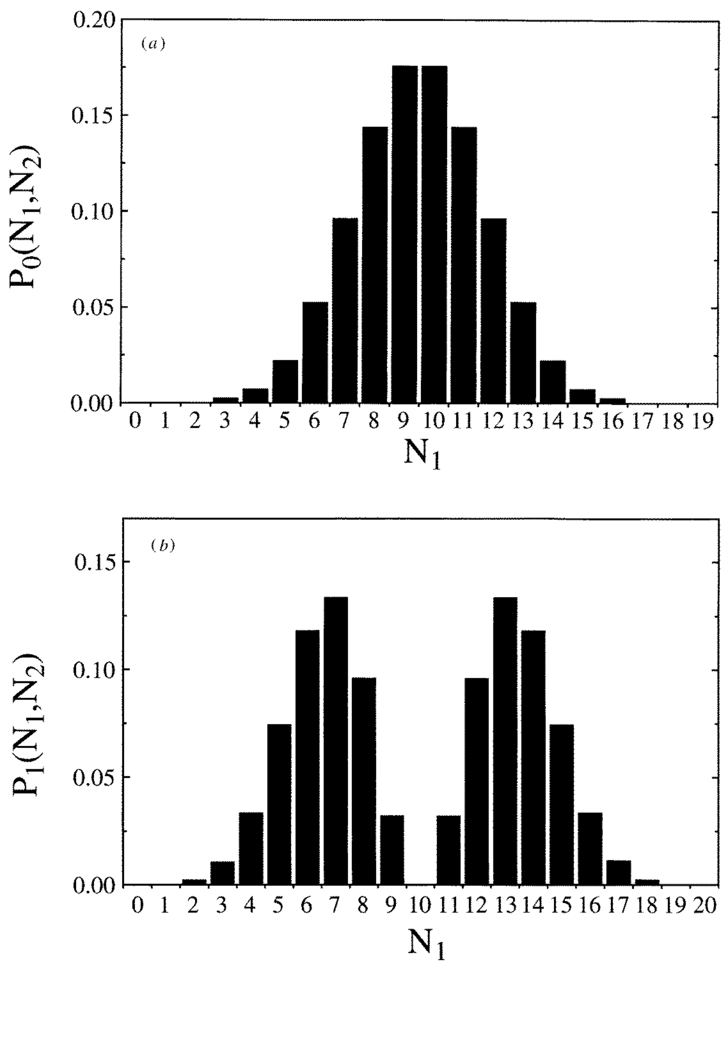

Figure 2. Output photon probability distribution for an N-photon state input at port 1 with (a) vacuum state or (b) single-photon state at port 2 ($N = 19$).

The above quantum probability cancellation effect due to a single-photon state is not strict with respect to the N-photon state. Since the effect is based on quantum interference between a single-photon state and an arbitrary N-photon state, it should exist for an arbitrary state which consists of an arbitrary number n of photons with probability $P_n^{\rm in}$. Our argument goes as follows. Since the vacuum state and the single-photon state are completely incoherent in the sense that they have a totally random phase distribution, the output fields due to the interference of one of these states with any other state will lose all the coherence information of the input. Therefore, the output photon distribution of the beamsplitter will depend only on the photon statistics $P_n^{\rm in}$ of the input state at port 1. So combining this fact with equations (1) and (4), we have the output photon distributions in the form of

$$P_0(N_1, N_2) = \frac{(N_1+N_2)!}{2^{N_1+N_2}N_1!N_2!} P^{\rm in}_{N_1+N_2} \tag{5a}$$

for vacuum input at port 2 and

$$P_1(N_1, N_2) = \frac{(N_1+N_2-1)!}{2^{N_1+N_2}N_1!N_2!} (N_1-N_2)^2 P^{\rm in}_{N_1+N_2-1} \tag{5b}$$

for a single-photon state input at port 2. Of course, we may rigorously derive the output photon distribution by following the procedure leading to (4). It can be shown that (5*a*) and (5*b*) are indeed the correct forms for the output photon distribution. Equations (5) are also a special case of equation (45) of [13].

By comparing (5*a*) and (5*b*), we easily find that $P_1(N_1 = N_2) = 0$ for a single-photon state input at port 2 while

$$P_0(N_1 = N_2) = \sum_{N_1=0}^{\infty} \frac{(2N_1)!}{2^{2N_1}(N_1!)^2} P^{\rm in}_{2N_1} \neq 0$$

for a vacuum input. Therefore, the effect of probability cancellation exists even for an arbitrary input state at port 1.

In an actual experiment, however, it is difficult to measure the complete distribution $P(N_1, N_2)$, but the distribution $P(N_1 - N_2 = M)$ can be measured by balanced homodyne detection [14]. From equations (5*a*) and (5*b*) we find that

$$\left.\begin{aligned} P_0(M) &= \sum_{N_1=M}^{\infty} \frac{(2N_1-M)!}{2^{2N_1-M}N_1!(N_1-M)!} P^{\rm in}_{2N_1-M} \\ P_1(M) &= M^2 \sum_{N_1=M}^{\infty} \frac{(2N_1-M-1)!}{2^{2N_1-M}N_1!(N_1-M)!} P^{\rm in}_{2N_1-M-1} \end{aligned}\right\} \quad \text{for } M \geqslant 0\,. \tag{6}$$

When $M < 0$, the symmetry between N_1, N_2 in (5*a*) and (5*b*) leads to $P(M) = P(-M)$.

Next, let us evaluate $P_0(M)$, $P_1(M)$ for some special states. For N-photon state input, we have $P_n^{\rm in} = \delta_{n,N}$, and (6) gives results similar to (1) and (4):

$$\begin{aligned} P_0(M) &= \frac{N!}{2^N(N/2+M/2)!(N/2-M/2)!} \\ &\approx \frac{2}{\sqrt{2N\pi}} {\rm e}^{-M^2/2N} \qquad \text{for } N \gg 1, M \\ P_1(M) &= \frac{M^2 N!}{2^{N+1}(N/2+M/2+1/2)!(N/2-M/2+1/2)!} \\ &\approx \frac{2}{\sqrt{2N\pi}} \frac{M^2}{N} {\rm e}^{-M^2/2N} \qquad \text{for } N \gg 1, M. \end{aligned} \tag{7}$$

The extra normalization factor of 2 in the approximated expressions in (7) is because $P(M) = 0$ for every other value of M. For coherent state, $P_n^{\text{in}} = \bar{n}^n \mathrm{e}^{-\bar{n}}/n!$ with $\bar{n}$ being the average photon number and we have from (5a) and (5b) that

$$P_0(M) = \mathrm{e}^{-\bar{n}} I_M(\bar{n}) \qquad P_1(M) = \frac{M^2}{\bar{n}} \mathrm{e}^{-\bar{n}} I_M(\bar{n}) \tag{8}$$

where $I_M(\bar{n})$ is the Bessel function with purely imaginary argument. Similar results as (8) were obtained in [10, 14]. For large $\bar{n}$,

$$P_0(M) \approx \frac{1}{\sqrt{2\bar{n}\pi}} \mathrm{e}^{-M^2/2\bar{n}} \qquad P_1(M) \approx \frac{1}{\sqrt{2\bar{n}\pi}} \frac{M^2}{\bar{n}} \mathrm{e}^{-M^2/2\bar{n}} \tag{9}$$

which has the same form as (7) for large N besides the factor of 2. This is not surprising if we consider the fact that when the photon number is large, the interference scheme discussed above becomes a homodyne detection scheme. Since both the vacuum state and the single-photon state have random phase distribution, homodyne detection with the N-photon state ($N \gg 1$) and the coherent state as local oscillators are equivalent. As a matter of fact, the output photon distributions will always have the form of (9) for any state as local oscillator, provided that its average photon number is large and the photon number fluctuation is much less than average photon number ($\sqrt{\langle \Delta n^2 \rangle} \ll \bar{n}$). We can see this point from (6): when $\sqrt{\langle \Delta n^2 \rangle} \ll \bar{n}$, P_n^{in} has a narrow peak around $\bar{n}$ and is a fast changing function as compared with other terms in the summation, therefore the contribution to the summation only comes from the few terms near $\bar{n}$, so that we can pull the slowly changing terms out of the sum, that is,

$$\begin{aligned} P_0(M) &\approx \frac{\bar{n}!}{2^{\bar{n}}(\bar{n}/2 - M)!(\bar{n}/2 + M)!} \sum_n P_n^{\text{in}}/2 \\ &\approx \frac{1}{\sqrt{2\bar{n}\pi}} \mathrm{e}^{-M^2/2\bar{n}} \qquad \text{when } \bar{n} \gg 1 \end{aligned} \tag{10a}$$

and similarly

$$P_1(M) \approx \frac{M^2/\bar{n}}{\sqrt{2\bar{n}\pi}} \mathrm{e}^{-M^2/2\bar{n}} \qquad \text{when } \bar{n} \gg 1\,. \tag{10b}$$

We can also understand this result from the fact that any fluctuation in the local oscillator is cancelled in the balanced homodyne detection scheme [15].

Furthermore, if we set $\bar{n} \to \infty$, we can replace the discrete variable M with a continuous one defined by $x = M/\sqrt{\bar{n}}$ and the probability distributions in (10a) and (10b) lead to probability densities of continuous variable x as

$$P_0(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2} \qquad P_1(x) = \frac{x^2}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2} \tag{11}$$

which correspond to the square of the absolute value of the wavefunction for the vacuum state and single-photon state, respectively. Thus by measuring $P(M)$ in homodyne detection, we can deduce the wavefunction of the input state at port 2 besides a phase factor which can be fixed by the technique of optical tomography [16, 17].

However, there is an exception to the above. It is well known that for thermal light,

$$\langle \Delta n^2 \rangle = \bar{n}(\bar{n} + 1)$$

so that $\sqrt{\langle \Delta n^2 \rangle} \approx \bar{n}$ and we cannot use the approximation in (10a) and (10b). For thermal light, $P_n^{\text{in}} = \bar{n}^n/(\bar{n}+1)^{n+1}$, so from (6), we have

$$\begin{aligned} P_0(M) &= q^M/\sqrt{2\bar{n}+1} \\ P_1(M) &= M q^M/\bar{n} \end{aligned} \qquad (M \geqslant 0) \tag{12}$$

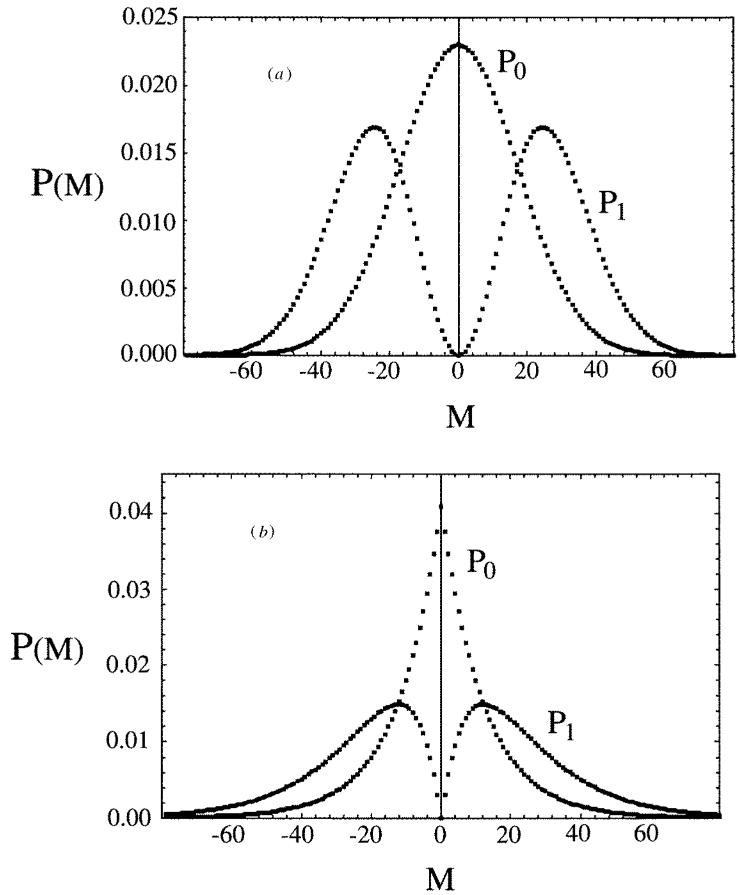

Figure 3. Probability distribution $P_{0,1}(M)$ for balanced homodyne detection of vacuum state and single-photon state with (*a*) coherent state or (*b*) thermal state as local oscillator. $\bar{n} = 300$.

with $q = 1 + 1/\bar{n} - \sqrt{2\bar{n}+1}/\bar{n}$. Therefore, the output photon distribution for thermal light input is different from that of coherent state input even when $\bar{n} \gg 1$. But the general trend in the change of the shape from $P_0(M)$ to $P_1(M)$ is similar in both states (figure 3) and the quantum interference effect due to a single photon is still there.

It is interesting to note that the quantum interference effect studied here has similarities with other types of intensity-independent interference effect where two interfering fields are substantially different in intensity [3, 4]. In both cases, the presence of the weak field can dramatically change the outcome of the result. However, the underlying principles are quite different in the two cases. Here all the N photons participate in the interference ($(N + 1)$-particle interference) whereas in [3, 4] only two photons are involved and the rest of the photons are not counted. Even though the non-classical field is weak here, the result is very non-classical in the sense that the probability of detecting equal intensities at the two outputs is zero ($P_1(M = 0) = 0$). It can be proved that in a similar situation (one field is weak and the other is strong), classical wave theory predicts that the probability is largest for equal intensity output at the two ports.

It should be noted that the quantum probability cancellation effect stems from the general principle of quantum interference, therefore it applies to arbitrary particles: bosons or fermions, particles without structure such as photons or with structures such as atoms.

Appendix. Derivation of equation (4)

First, because the beamsplitter is lossless, we immediately obtain the δ-function from photon number conservation. Therefore, we only need to calculate the case when $N_1+N_2 = N+1$. Because of this, there is no contribution from the exponential factors in (3) so that we can ignore them. Then we expand the expression inside the normal ordering operation in (3) and write them in terms of a_i $(i = 1, 2)$ as

$$\frac{1}{2^{N_1+N_2}N_1!N_2!}\sum_{\substack{m_1,m_2,m_3,m_4\\ n_1,n_2,n_3,n_4}}(-i)^{n_3+m_4}i^{m_3+n_4}\frac{N_1!N_2!(\hat{a}_1^{\dagger})^{k_1'}\hat{a}_1^{k_1}(\hat{a}_2^{\dagger})^{k_2'}\hat{a}_2^{k_2}}{m_1!m_2!m_3!m_4!n_1!n_2!n_3!n_4!} \tag{A1}$$

where $k_i' = m_i+m_3+n_i+n_3$, $k_i = m_i+m_4+n_i+n_4 (i = 1, 2)$ and $m_1+m_2+m_3+m_4 = N_1$, $n_1 + n_2 + n_3 + n_4 = N_2$. We write the operators in (A1) in normal ordering.

Since the quantum average is over the state $|\Phi\rangle = |N\rangle_1|1\rangle_2$, the average is non-zero only when $k_i' = k_i$ $(i = 1, 2)$ so that $m_3 + n_3 = m_4 + n_4$. Because mode 2 is in a single-photon state, this requires $k_2 = 0$, or 1. For the first choice, we must have $k_1 = N_1 + N_2 = N + 1$. But mode 1 is in the N-photon state, so its contribution is zero. For the second choice, we have the following possibilities:

(i) $m_2 = 1$ and $m_3 = m_4 = n_3 = n_4 = 0 = n_2$, $m_1 = N_1 - 1$ and $n_1 = N_2$ so that $k_1 = N_1 + N_2 - 1 = N$, for which the quantum average gives

$$\frac{N!}{2^{N_1+N_2}N_1!N_2!}\frac{N_1!N_2!}{(N_1-1)!N_2!} = \frac{N!}{2^{N+1}N_1!N_2!}N_1\,.$$

(ii) $n_2 = 1$ and $m_3 = m_4 = n_3 = n_4 = 0 = m_2$, $m_1 = N_1$ and $n_1 = N_2 - 1$ so that $k_1 = N_1 + N_2 - 1 = N$, for which the quantum average gives

$$\frac{N!}{2^{N_1+N_2}N_1!N_2!}\frac{N_1!N_2!}{N_1!(N_2-1)!} = \frac{N!}{2^{N+1}N_1!N_2!}N_2\,.$$

(iii) $m_3 = 1$, $m_4 = 1$ and $n_3 = n_4 = 0 = m_2 = n_2$, $m_1 = N_1 - 2$ and $n_1 = N_2$ so that $k_1 = N_1 + N_2 - 2 = N - 1$, for which the quantum average gives

$$\frac{N!}{2^{N_1+N_2}N_1!N_2!}\frac{N_1!N_2!}{(N_1-2)!N_2!} = \frac{N!}{2^{N+1}N_1!N_2!}N_1(N_1-1)\,.$$

(iv) $m_3 = 1$, $n_4 = 1$ and $n_3 = m_4 = 0 = m_2 = n_2$, $m_1 = N_1 - 1$ and $n_1 = N_2 - 1$ so that $k_1 = N_1 + N_2 - 2 = N - 1$, for which the quantum average gives

$$-\frac{N!}{2^{N_1+N_2}N_1!N_2!}\frac{N_1!N_2!}{(N_1-1)!(N_2-1)!} = -\frac{N!}{2^{N+1}N_1!N_2!}N_1N_2\,.$$

(v) $n_3 = 1$, $n_4 = 1$ and $m_3 = m_4 = 0 = m_2 = n_2$, $m_1 = N_1$ and $n_1 = N_2 - 2$ so that $k_1 = N_1 + N_2 - 2 = N - 1$, for which the quantum average gives

$$\frac{N!}{2^{N_1+N_2}N_1!N_2!}\frac{N_1!N_2!}{(N_1-2)!N_2!} = \frac{N!}{2^{N+1}N_1!N_2!}N_2(N_2-1)\,.$$

(vi) $n_3 = 1$, $m_4 = 1$ and $m_3 = n_4 = 0 = m_2 = n_2$, $m_1 = N_1 - 1$ and $n_1 = N_2 - 1$ so that $k_1 = N_1 + N_2 - 2 = N - 1$, for which the quantum average gives

$$-\frac{N!}{2^{N_1+N_2}N_1!N_2!}\frac{N_1!N_2!}{(N_1-1)!(N_2-1)!} = -\frac{N!}{2^{N+1}N_1!N_2!}N_1N_2\,.$$

Combine (1)–(6), then we obtain the probability $P_1(N_1, N_2)$ as

$$P_1(N_1, N_2) = \frac{N!}{2^{N+1}N_1!N_2!}(N_1 + N_2 + N_1(N_1 - 1) + N_2(N_2 - 1) - 2N_1N_2) \qquad \text{(A2)}$$

which is equation (4) after rearrangement.

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