

## Robustness of homodyne tomography to phase-insensitive noise

GIACOMO MAURO D'ARIANO<sup>†‡</sup> and NICOLETTA STERPI<sup>‡</sup>

<sup>†</sup> Department of Electrical and Computer Engineering, Department of Physics and Astronomy, Northwestern University, Evanston, IL 60208, USA

<sup>‡</sup> Dipartimento di Fisica 'A. Volta', via Bassi 6, 27100 Pavia, Italy

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**Abstract.** We study the effects of phase-insensitive noise on homodyne measurements of a radiation density matrix. We prove that this noise has an effect equivalent to a non-unit quantum efficiency at detectors. The overall effective quantum efficiency  $\eta_*$  of the measurement is evaluated in terms of the quantum efficiency at detectors and of the average number of noise photons added to the radiation field. For pure Gaussian-displacement noise, we show that half a photon of noise is enough to prevent the homodyne measurement of the density matrix.

### 1. Introduction

The problem of measuring the density matrix  $\hat{\rho}$  of radiation has been extensively considered both experimentally and theoretically [1, 2]. Homodyne tomography is presently the only viable method that can be used to achieve such measurement. This method is based on the idea that the density matrix can be evaluated in optical homodyne experiments from the collection of quadrature probability distributions for the radiation state. The field quadrature is defined as  $\hat{x}_\phi = [a^\dagger \exp(i\phi) + a \exp(-i\phi)]/2$ , where  $\phi$  is the phase shift with respect to the local oscillator ( $a^\dagger$  and  $a$  are the creation and annihilation operators of the field mode). As shown in [3], the matrix can be obtained after calculating the Wigner function as the inverse Radon transform of such quadrature distributions. By following the approach of [3], the feasibility of the matrix measurement was experimentally demonstrated [4, 5], even though the matrix evaluation from the Wigner function requires a filtering procedure of the experimental data that produces systematic errors. A direct method for measuring a radiation density matrix  $\hat{\rho}$  was proposed recently [6–8]: such a method avoids the intermediate calculation of the Wigner function because  $\hat{\rho}$  is written as an average over the quadrature probability distributions of an appropriate kernel operator. The density matrix elements are then directly evaluated as experimental averages on optical homodyne data of kernel functions (the matrix elements of the kernel operator). The kernel functions depend on the representation basis chosen for  $\hat{\rho}$  and on the detector's quantum efficiency  $\eta$ . In [7], depending on the matrix representation, the bounds for  $\eta$  have been established below which the matrix

elements cannot be measured. In particular, for  $\eta \leq 1/2$  the matrix cannot be measured in any known representation.

In this paper we demonstrate the robustness of the homodyne tomography method to phase-insensitive noise. The physical situations where such noise arises are common in many experiments: for example, phase-insensitive linear amplification, field damping towards a thermal distribution (due to losses along an optical fibre, or at a beam splitter), or linear interaction of the radiation field with random fluctuating classical fields. We show that the effect of such noise can be taken into account in the kernel operator by introducing an effective quantum efficiency  $\eta_*$  of the measurement. This is not surprising in the case of noise due to losses (see for example [9]), however it is not trivial for the case of amplification. The result of [7] is generalized in terms of  $\eta_*$ , i.e. if  $\eta_* \leq 1/2$  the density matrix cannot be measured in any known representation basis.

The present paper is organized as follows. The effect of phase-insensitive noise on radiation is described in section 2 by a Fokker Planck equation for the generalized Wigner function: in particular, this kind of noise arises for drift linear term in the field variables and constant diffusion coefficient. From the analytical solution of the Fokker Planck equation, we calculate the quadrature probability distributions as marginal distributions of the generalized Wigner function, and we evaluate the kernel operator, whose matrix elements are the kernel functions. We also introduce the quantum efficiency of the measurement  $\eta_*$  as a function of the average number of noise photons added to the radiation field and of homodyne detectors' quantum efficiency. In section 3 we focus our attention on pure Gaussian-displacement noise, corresponding to the case of zero field-gain. We show that this noise imposes very strong limits for  $\eta_*$ , discussing the bound for  $\eta_*$  in order to measure the density matrix. Section 4 concludes the paper.

## 2. The kernel operator

The effect of additive phase-insensitive noise on the density matrix is determined by the following Fokker Planck equation for the generalized Wigner function  $W_s(\alpha, \bar{\alpha}; t)$  (for ordering parameter  $s$ )

$$\partial_t W_s(\alpha, \bar{\alpha}; t) = [Q(\partial_\alpha \alpha + \partial_{\bar{\alpha}} \bar{\alpha}) + 2D_s \partial_{\alpha, \bar{\alpha}}^2] W_s(\alpha, \bar{\alpha}; t), \quad (1)$$

where  $Q$  and  $D_s$  are, respectively, the drift and diffusion coefficients. Equation (1) corresponds to the master equation

$$\partial_t \hat{\rho}_t = 2[A L[a^\dagger] + B L[a]] \hat{\rho}_t, \quad (2)$$

for  $Q = B - A$  and  $2D_s = A + B + s(A - B)$ . In equation (2),  $L[\hat{c}]$  denotes the Lindblad superoperator [10] (affecting the radiation state as  $L[\hat{c}] \hat{\rho} \doteq \hat{c} \hat{\rho} \hat{c}^\dagger - \frac{1}{2} \{ \hat{c}^\dagger \hat{c}, \hat{\rho} \}$ ). Since  $L[a \exp(-i\phi)] = L[a]$ , the dynamical evolution for the density matrix is phase-insensitive.

In the present paper  $\hat{\rho} \equiv \hat{\rho}_0$  denotes the state of radiation that must be measured, whereas the 'time-evolved' operator  $\hat{\rho}_t$  denotes the state affected by noise.

The evolution of the average field is given by  $\langle a \rangle_t = g \langle a \rangle_0$ , with  $g = \exp(-Qt)$ . Thus, one can see that for  $A > B$  equation (2) describes phase-insensitive amplification with field gain  $g$ . In this case, equation (2) models unsaturated

parametric amplification with thermal idler mode, or unsaturated laser action (if  $A$  and  $B$  are proportional to atomic populations on the upper and lower lasing levels, respectively). For  $B > A$ , on the other hand, equation (2) describes, for example, a field damping towards the thermal distribution (with inverse photon lifetime  $\Gamma = 2(B - A)$  and equilibrium photon number  $\bar{m} = A/(B - A)$ ), a loss along an optical fibre or a beam splitter, or even a loss due to frequency conversion [11]. The case  $A = B$  leaves the average field invariant, but introduces noise, changing the average photon number as  $\langle a^\dagger a \rangle_t = \langle a^\dagger a \rangle_0 + \bar{n}$ , where  $\bar{n} = 2At$ . In this case the solution of equation (2) is given by

$$\hat{\rho}_t = \int d^2\beta \frac{1}{\pi\bar{n}} \exp\left(-\frac{|\beta|^2}{\bar{n}}\right) \hat{D}(\beta) \hat{\rho} \hat{D}^\dagger(\beta), \tag{3}$$

where  $\hat{D}(\beta) = \exp(\beta a^\dagger - \bar{\beta} a)$  is the usual displacement operator. Equation (3) describes the Gaussian-displacement noise studied in [12] and [13], also commonly referred to as ‘thermal noise’. This noise models many kinds of undesired environmental effects, as, for example, the linear interaction of the signal field with random fluctuating classical fields, and, obviously, zero-gain phase-insensitive linear amplification.

The solution of equation (1) is the Gaussian convolution

$$W_s(\alpha, \bar{\alpha}; t) = \int \frac{d^2\beta}{\pi\delta_s^2} \exp\left[-\frac{|\alpha - g\beta|^2}{\delta_s^2}\right] W_s(\beta, \bar{\beta}; 0), \tag{4}$$

with

$$\delta_s^2 = \frac{D_s}{Q} (1 - g^2) \quad g = \exp(-Qt). \tag{5}$$

The probability distribution  $p_n(x, \phi; t)$  of the field quadrature is the marginal distribution of the Wigner function, as [3]

$$p_\eta(x, \phi; t) = \int_{-\infty}^{+\infty} dy W_{1-\eta^{-1}}((x + iy) \exp(i\phi), (x - iy) \exp(-i\phi); t), \tag{6}$$

where  $\eta$  is the quantum efficiency of homodyne detectors,  $s = 1 - \eta^{-1}$  and  $\alpha = x + iy$ . Thus, from equation (4) one obtains the quadrature probability in the form of a Gaussian convolution, as

$$p_\eta(x, \phi; t) = \exp(Qt) \int_{-\infty}^{\infty} \frac{dx'}{(2\pi\Delta_\eta^2)^{1/2}} \exp\left[-\frac{(x' - g^{-1}x)^2}{2\Delta_\eta^2}\right] p_\eta(x', \phi; 0), \tag{7}$$

with  $\Delta_\eta^2 = \frac{1}{2} g^{-2} \delta_{1-\eta^{-1}}^2$ . The analytical solution for the quadrature probability distribution (7) allows calculation of the kernel operator, that is needed to measure the density matrix  $\hat{\rho}$ . Indeed, we note that from equation (7) the generating function of the  $\hat{x}_\phi$ -moments for the initial ideal probability  $p(x, \phi; 0)$  can be written in terms of the experimental one as follows

$$\int_{-\infty}^{+\infty} dx p(x, \phi; 0) \exp(igkx) = \exp\left(\frac{\Delta_\eta^2}{2} g^2 k^2 + \frac{1-\eta}{8\eta} g^2 k_2\right) \int_{-\infty}^{+\infty} dx p_\eta(x, \phi; t) \exp(ikx). \tag{8}$$

Thus, the operator identity [7]

$$\hat{\rho} = \int_0^\pi \frac{d\phi}{\pi} \int_{-\infty}^{+\infty} dk \frac{|k|}{4} \text{Tr} [\hat{\rho} \exp(ik\hat{x}_\phi)] \exp(-ik\hat{x}_\phi), \quad (9)$$

is written as

$$\hat{\rho} = \int_0^\pi \frac{d\phi}{\pi} \int_{-\infty}^{+\infty} dx p_\eta(x, \phi; t) K_{\eta^*}(g^{-1}x - \hat{x}_\phi), \quad (10)$$

with kernel operator

$$K_{\eta^*}(g^{-1}x - \hat{x}_\phi) = \int_{-\infty}^{+\infty} dk \frac{|k|}{4} \exp\left(\frac{1 - \eta^*}{8\eta^*} k^2\right) \exp[ik(g^{-1}x - \hat{x}_\phi)] \quad (11)$$

and overall effective quantum efficiency  $\eta^*$  defined by the relation

$$\eta_*^{-1} = \eta^{-1} + 4\Delta_\eta^2. \quad (12)$$

The efficiency  $\eta^*$  is written in terms of master equation parameters as

$$\eta_*^{-1} = \exp[2(B - A)t] \eta^{-1} + \frac{2A}{B - A} \{\exp[2(B - A)t] - 1\}, \quad (13)$$

whereas in terms of the field gain  $g$  and of the average number of photons it reads

$$\eta_*^{-1} = \eta^{-1} + g^{-2} [2\langle a^\dagger a \rangle_t + \eta^{-1}] - [2\langle a^\dagger a \rangle_0 + \eta^{-1}]. \quad (14)$$

In the case of pure Gaussian-displacement noise ( $Q = 0$  in equation (1), i.e.  $g = 1$ ), the density matrix undergoes the transformation (3) and the overall effective quantum efficiency is given by

$$\eta_*^{-1} = \eta^{-1} + 2\bar{n}. \quad (15)$$

As we will see in the next section, the conditions for the density matrix measurement require that  $\eta_* > 1/2$ . Thus, it is just sufficient to have half a photon of Gaussian noise to completely prevent the homodyne measurement of the matrix.

### 3. Feasibility of the measurement

Equation (10) shows that any density matrix element  $\langle \psi | \hat{\rho} | \varphi \rangle$  is the expectation value of the kernel function

$$f_{\eta^*}^{\psi\phi}(g^{-1}x; \phi) = \langle \psi | K_{\eta^*}(g^{-1}x - \hat{x}_\phi) | \varphi \rangle \quad (16)$$

for quadrature probability distribution  $p_\eta(x, \phi; t)$  (where we denote any pair of basis vectors of the Fock space by  $|\psi\rangle$  and  $|\varphi\rangle$ ). Finally, we have

$$\langle \psi | \hat{\rho} | \varphi \rangle = \int_0^\pi \frac{d\phi}{\pi} \int_{-\infty}^{+\infty} dx p_\eta(x, \phi; t) f_{\eta^*}^{\psi\phi}(g^{-1}x; \phi). \quad (17)$$

The matrix element is measured as follows. By means of homodyne detection, the field quadrature  $\hat{x}_\phi$  is measured at any desired phase shift  $\phi$  with respect to the local oscillator. Then  $\langle \psi | \hat{\rho} | \varphi \rangle$  is evaluated by averaging the kernel function, which is calculated for (scaled) random homodyne outcomes: as the experimental data are distributed according to the probability  $p_\eta(x, \phi; t)$ , this average gives a measurement of the matrix element. In other words,  $\langle \psi | \hat{\rho} | \varphi \rangle$  is measured by experimentally sampling the kernel function.

The present method can be used if the kernel function is bounded for any  $\phi \in [0, \pi[$ : indeed, in this case the central limit theorem guarantees that  $\langle \psi | \hat{\rho} | \phi \rangle$  can be sampled over a sufficiently large set of data, because the different values obtained from different experiments are Gaussian distributed and, moreover, the confidence interval of the measured matrix element can be evaluated.

Hereafter we will focus our attention on pure Gaussian-displacement noise. In this case the corresponding kernel operator reads

$$K_{\eta_*}(x - \hat{x}_\phi) = \int_{-\infty}^{+\infty} dk \frac{|k|}{4} \exp\left(\frac{1 - \eta_*}{8\eta_*} k^2 + ikx\right) \exp(-ik\hat{x}_\phi) \quad (18)$$

with effective quantum efficiency given by equation (15). For  $g = 1$ , equation (18) shows that the kernel function (16) is bounded if  $\langle \psi | \exp(-ik\hat{x}_\phi) | \phi \rangle$  decays faster than  $\exp[-k^2(1 - \eta_*)/8\eta_*]$ . By following the same lines of [7], we introduce the ‘resolution’  $\epsilon_\psi^2(\phi)$  of vector  $|\psi\rangle$  in the representation of quadrature eigenstates by the relation

$$|_\phi \langle x | \psi \rangle|^2 \simeq \exp\left(-\frac{x^2}{2\epsilon_\psi^2(\phi)}\right), \quad (19)$$

where the symbol  $\simeq$  denotes the leading term as a function of  $x$ . Then we evaluate the convergence of the matrix element in equation (16). We conclude that the kernel function is bounded if (for any  $\phi \in [0, \pi[$ )

$$\eta_* > \frac{1}{1 + 4\epsilon^2(\phi)}, \quad (20)$$

where we introduced the harmonic mean  $\epsilon^2(\phi)$  of the ‘resolutions’  $\epsilon_\psi^2(\phi)$  and  $\epsilon_\phi^2(\phi)$  as

$$\frac{2}{\epsilon^2(\phi)} = \frac{1}{\epsilon_\psi^2(\phi)} + \frac{1}{\epsilon_\phi^2(\phi)}. \quad (21)$$

If the inequality (20) is satisfied, the matrix element  $\langle \psi | \hat{\rho} | \phi \rangle$  can be measured. In other words,  $\langle \psi | \hat{\rho} | \phi \rangle$  can be measured if the harmonic mean of the resolutions for vectors  $|\psi\rangle$  and  $|\phi\rangle$  in the quadrature representations satisfies the bound

$$\min_{\phi \in [0, \pi[} \epsilon(\phi) > \frac{1}{2} \left( \frac{1 - \eta_*}{\eta_*} \right)^{1/2}. \quad (22)$$

In general, from equation (20) one has that for any representation satisfying the relation

$$\epsilon(\phi) \epsilon\left(\phi + \frac{\pi}{2}\right) = \frac{1}{4} \quad (23)$$

the density matrix cannot be measured if  $\eta_* \leq 1/2$  (see also [14]). Equation (23) is satisfied by all customary representations, including number, coherent and squeezed representations.

The existence of such a lower bound for  $\eta_*$  is of fundamental relevance, as it prevents one from measuring the wave function of a single system by schemes of repeated weak indirect measurements on the same system [15]. One might wonder if it is possible to find a basis for the Fock space with the product of resolutions in equation (23) larger than  $1/4$ . This point has been discussed in [2] where, in particular, it was noted that the product of resolutions is preserved by a unitary

transformation of the basis, so that it is very difficult to devise an ‘exotic’ representation with product of resolutions larger than  $1/4$ .

#### 4. Conclusions

In this paper we demonstrated that the method of homodyne tomography can be used even if phase-insensitive noise affects the radiation state. We introduced a linear Fokker Planck equation for the generalized Wigner function to describe this noise. We evaluated the kernel operator, whose matrix elements are the kernel functions that must be averaged on experimental homodyne data in order to obtain the density matrix elements. The effects of such noise are taken into account by introducing an overall effective quantum efficiency  $\eta_*$  of the measurement that is a function of the homodyne detector’s quantum efficiency and of the average number of noise photons added to the radiation field. Thus the lower bounds for the quantum efficiency given in [7] are rewritten in terms of  $\eta_*$ . In particular, the detrimental effect of the noise for gain  $g = 1$  (Gaussian noise) is very strong. Indeed, it is sufficient to have just half a photon of noise to prevent the homodyne measurement of the density matrix.

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