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Optical Communication with Two-Photon Coherent States—Part III: Quantum Measurements Realizable with Photoemissive Detectors

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***Abstract*—In Part I of this three-part study it was shown that the use of two-photon coherent state (TCS) radiation may yield significant performance gains in free-space optical communication if the receiver makes a quantum measurement of a single field quadrature. In Part II it was shown that homodyne detection achieves the same signal-to-noise ratio as the quantum field quadrature measurement, thus providing a receiver which realizes the linear modulation TCS performance gain found in Part I. Furthermore, it was shown in Part II that if homodyne detection does exactly correspond to the field quadrature measurement, then a large binary communication performance gain is afforded by homodyne detection of antipodal TCS signals. The full equivalence of homodyne detection and single-quadrature field measurement, as well as that of heterodyne detection and two-quadrature field measurement, is established. Furthermore, a heterodyne configuration which uses a TCS image-band oscillator in addition to the usual coherent state local oscillator is studied. This configuration, termed TCS heterodyne detection, is shown to realize all the quantum measurements described by arbitrary TCS. The foregoing results are obtained by means of a representation theorem which shows that photoemissive detection realizes the photon flux density measurement.**

Manuscript received November 13, 1978; revised April 6, 1979. This work was supported by the National Aeronautics and Space Administration under Grant NGL 22-009-013. This paper was presented at the IEEE International Symposium on Information Theory, Grignano, Italy, June 1979.

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I. INTRODUCTION

THE WORK to date in quantum communication theory has focused primarily on determining optimum quantum measurements for various detection and estimation problems and evaluating the resulting system perfor-

0018-9448/80/0100-0078$00.75

mance [1]–[6]. In these studies it is generally assumed that the radiation field in the vicinity of the receiver is in a Glauber coherent state or classically random mixture of such states. This is a realistic assumption in that ordinary light sources such as incandescent lamps, light-emitting diodes, and lasers produce radiation fields which satisfy the foregoing condition. Moreover, under this assumption the conditional Poisson process model for direct detection and the signal plus additive white Guassian noise models for heterodyne and homodyne detection are quantum mechanically correct [7], [8], so that optimum quantum performance may easily be compared with that of conventional receivers. However, when the optium quantum receiver outperforms the conventional techniques (i.e., when the required quantum measurement is not one of those realized by direct detection, heterodyne detection, or homodyne detection) very little has been said about how to achieve the quantum performance limit. Kennedy [9] found a receiver for the detection of binary coherent-state signals whose error probability has the same exponential behavior as the optimum quantum receiver; Kennedy's receiver uses a homodyne-like configuration with a "weak" local oscillator that is matched to one of the signal waveforms. Dolinar [10], [11] extended Kennedy's results by allowing the local oscillator to depend in a casual manner on the point process output of the photodetector; using this structure he obtained an explicit realization of the optimum quantum receiver for binary coherent-state signals.

The work of Kennedy and Dolinar suggests the need for a comprehensive assessment of the quantum measurements that can be realized with photoemissive detectors and ancillary apparatus (local oscillators, beamsplitters, etc.). It has long been known that photoemissive detection of a multimode radiation field with a unity quantum efficiency detector yields a random variable output whose statistics are the same as those predicted for quantum measurement of the total photon number operator for the field [7], [12]. This result, which is valid for any quantum state of the radiation field, is taken to mean that unity quantum efficiency direct detection realizes the number operator measurement.[1] No similar quantum description is as yet available for homodyne or heterodyne detection, although it has been accepted without a general proof that homodyning corresponds to the quantum measurement of a single field quadrature and heterodyning to that of both field quadratures. Personick [8] used the conditional Poisson model for direct detection to produce an explicit demonstration of the foregoing assertions for coherent-state input fields. However, because the conditional Poisson model is not valid for arbitrary quantum states of the input field [7], the complete quantum statistics of homodyne and heterodyne detection remain to be found.

[1] Throughout this paper we shall regard an apparatus whose classical random variable output has the same statistics as a particular quantum measurement (for *arbitrary* states of the radiation field) to be a realization of that measurement.

Our interest in the realization of quantum measurements with photoemissive detectors has evolved from our work in quantum transmitter theory. In [13] we investigated the use of movel radiation states, called two-photon coherent states [14] (TCS), in free-space optical communications. These states do not possess well-behaved diagonal P-representations, i.e., they cannot be interpreted as a classical mixture of Glauber coherent states. We showed in [13] that TCS radiation maximizes the field measurement signal-to-noise ratio in free-space information transmission via linear modulation; an optimum TCS system has a substantial performance advantage over a coherent-state system. In [15] we investigated the use of conventional reception techniques with TCS radiation. We demonstrated there that in a lossless limit the signal-to-noise ratios of homodyne and heterodyne detection of a single-mode radiation field in an arbitrary quantum state coincide with those of the abstract quantum measurements of $a_1 \equiv (a+a^\dagger)/2$ and a, respectively, where a is the photon annihilation operator of the nonvacuum field mode. This implied that homodyne detection achieves the full TCS linear modulation signal-to-noise ratio improvement predicted, with the assumption that a_1 is measured in [13]. We further showed in [15] that if homodyning indeed realizes the abstract a_1 measurement for TCS inputs, then the use of binary antipodal TCS signals with homodyne reception and a minimum error probability decision rule leads to an enormous near-field energy advantage compared to the similar coherent-state system. Thus the quantum statistics of homodyne detection are a vital issue in TCS communication theory, as well as an important general result in measurement realization.

In this final installment of our three-part study of TCS communication theory we shall address the quantum measurement realization issues previously described. Specifically, we shall prove the full equivalence of homodyne detection to single-quadrature field measurement and that of heterodyne detection to two-quadrature field measurement. These results will be obtained from a representation theorem that identifies the statistics of the photon flux density measurement with those of the point process output of a photoemissive detector. We believe that this representation theorem will form the foundation for an exhaustive evaluation of the measurements realizable with photodetectors. Toward that end, we will show that a new receiver configuration, which we call TCS heterodyning, can be used to realize all the measurements described by TCS. This class of measurements is identical to the canonical measurements of Holevo [3], [16] apart from an additive classical noise.

In Section II we begin our study by establishing notation and two key technical lemmas. In Section III we prove the operator representation theorem for photodetection and illustrate the direct utility of the result in actual computations. These results are applied in Sections IV and V, respectively, to homodyne and TCS-heterodyne detection; ordinary heterodyne detection is treated as a special case of TCS-heterodyne detection. Finally, in Section VI we will illustrate the utility of our results.

II. Preliminaries

We have collected in this section the notation and two calculational lemmas that will be employed repeatedly in Sections III–V.

A. Quantum-Optics Notation

The development in this paper will be carried out for fields with a finite number N of spatio-temporal modes. Formally, all our results extend to the case of an infinite number of modes, although in actual applications there is usually some finite number of modes which suffices.[2] Let $\{a_n: 1 \leq n \leq N\}$ be the modal photon annihilation operators associated with the N field modes under consideration. The $\{a_n\}$ obey the canonical commutation rules (CCR)

$$[a_n, a_m] = 0, \qquad [a_n, a_m^\dagger] = \delta_{nm}, \tag{2.1}$$

where $a_n^\dagger$, the adjoint of a_n, is the creation operator associated with the nth mode. We use $\boldsymbol{a}$ and $\boldsymbol{a}^\dagger$ to denote column vectors whose nth elements are a_n and $a_n^\dagger$, respectively; $\boldsymbol{a}^T$ and $(\boldsymbol{a}^\dagger)^T$ denote the row vectors obtained by transposition. For c-numbers $\{\alpha_n: 1 \leq n \leq N\}$ we use $\boldsymbol{\alpha}$ to denote the column vector whose nth element is α_n, and $\boldsymbol{\alpha}^\dagger$ to denote the row vector whose nth element is α_n^*.

We shall need to deal with operators, $M(\boldsymbol{a}^\dagger, \boldsymbol{a})$, which are functions of $\boldsymbol{a}$ and $\boldsymbol{a}^\dagger$. Such operators have normally ordered representations [18], [19]

$$M^{(n)}(\boldsymbol{\alpha}^*, \boldsymbol{\alpha}) \equiv \langle \boldsymbol{\alpha} | M(\boldsymbol{a}^\dagger, \boldsymbol{a}) | \boldsymbol{\alpha} \rangle \tag{2.2}$$

in terms of the product of modal coherent states $|\boldsymbol{\alpha}\rangle \equiv \otimes_n |\alpha_n\rangle$, where $a_n|\alpha_n\rangle = \alpha_n|\alpha_n\rangle$. The operator M is uniquely determined by $M^{(n)}$ [14], [19], or equivalently, by the Fourier transform of $M^{(n)}$

$$\begin{aligned}\chi_A^M(\boldsymbol{\zeta}^*, \boldsymbol{\zeta}) &= \int M^{(n)}(\boldsymbol{\alpha}^*, \boldsymbol{\alpha}) \exp(\boldsymbol{\zeta}^T \boldsymbol{\alpha}^* - \boldsymbol{\alpha}^T \boldsymbol{\zeta}^*)\, d^2\boldsymbol{\alpha}/\pi^N \\ &= \operatorname{tr}\left[M \exp(-\boldsymbol{\zeta}^\dagger \boldsymbol{a}) \exp(\boldsymbol{\zeta}^T \boldsymbol{a}^\dagger)\right],\end{aligned} \tag{2.3}$$

if it exists.

For the particular case of the density operator ρ for the field modes associated with $\boldsymbol{a}$, $\rho^{(n)}(\boldsymbol{\alpha}^*, \boldsymbol{\alpha})/\pi^N$ is a probability density [18]–[20], so that its Fourier transform $\chi_A^\rho(\boldsymbol{\zeta}^*, \boldsymbol{\zeta})$ (called the antinormally ordered characteristic function) is always well defined. The Wigner distribution for ρ also exists [18]–[20], viz.,

$$W^\rho(\boldsymbol{\alpha}^*, \boldsymbol{\alpha}) \equiv \int \chi_W^\rho(\boldsymbol{\zeta}^*, \boldsymbol{\zeta}) \exp(\boldsymbol{\alpha}^T \boldsymbol{\zeta}^* - \boldsymbol{\zeta}^T \boldsymbol{\alpha}^*)\, d^2\boldsymbol{\zeta}/\pi^{2N}, \tag{2.4}$$

where

$$\begin{aligned}\chi_W^\rho(\boldsymbol{\zeta}^*, \boldsymbol{\zeta}) &\equiv \operatorname{tr}\left[\rho \exp(-\boldsymbol{\zeta}^\dagger \boldsymbol{a} + \boldsymbol{\zeta}^T \boldsymbol{a}^\dagger)\right] \\ &= \exp(|\boldsymbol{\zeta}|^2/2) \chi_A^\rho(\boldsymbol{\zeta}^*, \boldsymbol{\zeta}).\end{aligned} \tag{2.5}$$

[2] To limit our treatment to single-mode fields, however, would be unduly restrictive as they are necessarily first-order coherent [17], i.e., coherent in space and time.

For a general operator M, $\chi_W^M(\boldsymbol{\zeta}^*, \boldsymbol{\zeta})$ defined as in (2.5) may exist, and if both χ_A^M and χ_W^M exist they are related as shown above for $M = \rho$. Furthermore, if the appropriate χ^M functions exist, we can compute the expectation value of M, $\operatorname{tr}(\rho M)$, via [14]:

$$\operatorname{tr}(\rho M) = \int \chi_A^M(-\boldsymbol{\zeta}^*, -\boldsymbol{\zeta}) \chi_A^\rho(\boldsymbol{\zeta}^*, \boldsymbol{\zeta}) e^{|\boldsymbol{\zeta}|^2} d^2\boldsymbol{\zeta}/\pi^N \tag{2.6a}$$

or

$$\operatorname{tr}(\rho M) = \int \chi_W^M(-\boldsymbol{\zeta}^*, -\boldsymbol{\zeta}) \chi_W^\rho(\boldsymbol{\zeta}^*, \boldsymbol{\zeta})\, d^2\boldsymbol{\zeta}/\pi^N. \tag{2.6b}$$

Equations (2.6a) and (2.6b)[3] will be used for calculations in the sequel in place of the more familiar expressions involving the P-representation, because the latter can be extremely singular. In particular, χ_A^ρ and χ_W^ρ exist for any density operator, whereas TCS radiation does not possess a P-representation in terms of coherent states $|\boldsymbol{\alpha}\rangle$ in any useful sense.

B. Two-Photon Coherent States

A comprehensive discussion of the mathematical and physical nature of TCS radiation has been given in [14], and a brief summary of the TCS characteristics relevant to communication appears in [13]. Our purpose here is to redefine the notation used for a TCS in a way that facilitates later analyses and to present, using this new notation, some characteristic function results that will be needed in the sequel.

For a single mode of the radiation field with photon annihilation operator a the TCS $|\beta\rangle_g$ are the eigenstates of a transformed photon operator, i.e.,

$$b|\beta\rangle_g = \beta|\beta\rangle_g, \tag{2.7}$$

where

$$b = \mu a + \nu a^\dagger \tag{2.8}$$

for complex μ and ν such that

$$|\mu|^2 - |\nu|^2 = 1. \tag{2.9}$$

We shall use $|\mu\nu\alpha\rangle$ to denote the TCS $|\beta\rangle_g$ associated with parameters μ and ν, where $\alpha \equiv \mu^*\beta - \nu\beta^*$. This implies that (2.7) becomes

$$b|\mu\nu\alpha\rangle = (\mu\alpha + \nu\alpha^*)|\mu\nu\alpha\rangle \tag{2.7a}$$

in the new notation, with (2.8) and (2.9) as before. Using the $|\mu\nu\alpha\rangle$ notation we have

$$\int |\mu\nu\alpha\rangle\langle\mu\nu\alpha|\, d^2\alpha/\pi = I \tag{2.10}$$

and

$$a = \int \alpha |\mu\nu\alpha\rangle\langle\mu\nu\alpha|\, d^2\alpha/\pi, \tag{2.11}$$

as would be the case for coherent states $|\alpha\rangle$, but

$$aa^\dagger = \int |\alpha|^2 |\mu\nu\alpha\rangle\langle\mu\nu\alpha|\, d^2\alpha/\pi - |\nu|^2 \tag{2.12}$$

[3] There is an error in [14, eq. (2.26)]; the $\chi_A^M(\eta^*, \eta)$ term should not be conjugated, but should appear as $\chi_A^M(-\eta^*, -\eta)$.

and

$$a^2 = \int \alpha^2 |\mu\nu\alpha\rangle\langle\mu\nu\alpha| d^2\alpha/\pi + \mu^*\nu, \qquad (2.13)$$

etc., unlike the coherent-state representations.

For an N-mode radiation field the general form of a TCS is quite complicated. It will be sufficient for our present study of quantum-measurement characterization to restrict our attention to a product TCS of the form $|\mu\nu\boldsymbol{\alpha}\rangle \equiv \otimes_n |\mu\nu\alpha_n\rangle$. In this case we have that [14]

$$\chi_A^\rho(\boldsymbol{\zeta}^*, \boldsymbol{\zeta}) = \exp\left[-|\mu|^2|\boldsymbol{\zeta}|^2 - \mathrm{Re}(\boldsymbol{\zeta}^\dagger\boldsymbol{\zeta}^*\mu^*\nu) - \boldsymbol{\zeta}^\dagger\boldsymbol{\alpha} + \boldsymbol{\zeta}^T\boldsymbol{\alpha}^*\right] \qquad (2.14)$$

is the antinormally ordered characteristic function for the density operator $\rho = |\mu\nu\boldsymbol{\alpha}\rangle\langle\mu\nu\boldsymbol{\alpha}|$.

For an arbitrary N-mode density operator ρ the probability density $\rho^{(n)}(\boldsymbol{\alpha}^*, \boldsymbol{\alpha})/\pi^N = \langle\boldsymbol{\alpha}|\rho|\boldsymbol{\alpha}\rangle/\pi^N$ (or, equivalently, the associated characteristic function $\chi_A^\rho(\boldsymbol{\zeta}^*, \boldsymbol{\zeta})$) directly gives the statistics of measurements characterized by $|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|$, i.e., by the probability operator measure (POM) implied by the resolution of the identity [21]

$$\int |\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}| d^2\boldsymbol{\alpha}/\pi^N = I. \qquad (2.15)$$

In a similar manner we may assert that $\langle\mu\nu\boldsymbol{\alpha}|\rho|\mu\nu\boldsymbol{\alpha}\rangle/\pi^N$ or its characteristic function

$$\chi_T^\rho(\boldsymbol{\zeta}^*, \boldsymbol{\zeta}) \equiv \int \langle\mu\nu\boldsymbol{\alpha}|\rho|\mu\nu\boldsymbol{\alpha}\rangle \exp(\boldsymbol{\zeta}^T\boldsymbol{\alpha}^* - \boldsymbol{\alpha}^T\boldsymbol{\zeta}^*) d^2\boldsymbol{\alpha}/\pi^N \qquad (2.16)$$

gives the statistics of measurements characterized by $|\mu\nu\boldsymbol{\alpha}\rangle\langle\mu\nu\boldsymbol{\alpha}|$, i.e., by the POM implied by the N-mode version of (2.10). From (2.7)–(2.11) it is straightforward to show that

$$\chi_T^\rho(\boldsymbol{\zeta}^*, \boldsymbol{\zeta}) = \chi_A^\rho(\boldsymbol{\zeta}^*, \boldsymbol{\zeta}) \exp\left[-|\nu|^2|\boldsymbol{\zeta}|^2 - \mathrm{Re}(\boldsymbol{\zeta}^\dagger\boldsymbol{\zeta}^*\mu^*\nu)\right]. \qquad (2.17)$$

Note that the addition of an independent classical noise to the modes associated with $\boldsymbol{a}$ merely multiplies the quantum characteristic functions χ_A^ρ of χ_X^ρ by the characteristic function of the classical noise. The class of states obtained by addition of an independent Guassian noise to general multimode TCS correspond to the Gaussian states discussed by Holevo [3], [16]. In this paper we will limit ourselves to measurements described by Guassian states of the form given by (2.14) or its modification, to include another positive semidefinite quadratic form in $(\boldsymbol{\zeta}^*, \boldsymbol{\zeta})$ in the exponent representing an independent additive Guassian noise. The general case can be treated by the same method but first requires a considerable development of multimode TCS calculus which will be presented elsewhere.

C. Technical Lemmas

In the succeeding sections we will need to put an operator of Guassian form

$$G = \exp\left[(\boldsymbol{a}^\dagger)^T A \boldsymbol{a}\right], \qquad (2.18)$$

where A is an arbitrary complex matrix, into normal order. This is accomplished in the following lemma.

Lemma 1: For G defined by (2.18) we have

$$G^{(n)}(\boldsymbol{\alpha}^*, \boldsymbol{\alpha}) = \exp\left[\boldsymbol{\alpha}^\dagger(e^A - I)\boldsymbol{\alpha}\right]. \qquad (2.19)$$

Remarks: The one-dimensional (single-mode) limit of Lemma 1 is well-known in quantum optics [18]. The multimode result we require can be proved by a differential-equation technique [14], [18], but it also follows from [22, eq. (4.24)]. We omit the proof.

The following multidimensional Guassian integral will be needed in subsequent developments; a proof is provided here because the usual result assumes that A is Hermitian [23, p. 99].

Lemma 2: Let A be an arbitrary complex-valued matrix such that $A + A^\dagger$ is positive definite, then

$$\int \exp\left[-\boldsymbol{\alpha}^\dagger A\boldsymbol{\alpha} - \boldsymbol{\zeta}^\dagger\boldsymbol{\alpha} + \boldsymbol{\zeta}^T\boldsymbol{\alpha}^*\right] d^2\boldsymbol{\alpha}/\pi^N = (\det A)^{-1}\exp\left[-\boldsymbol{\zeta}^\dagger A^{-1}\boldsymbol{\zeta}\right]. \qquad (2.20)$$

Proof: It is well-known that for $\boldsymbol{x}$ a real-valued p-vector, $\boldsymbol{f}$ a complex-valued p-vector, and A' a complex-valued *symmetric* matrix whose real part is positive definite, that [23, p. 98], [24]

$$\int \exp\left[-\boldsymbol{x}^T A' \boldsymbol{x} + \boldsymbol{f}^T\boldsymbol{x}\right] d\boldsymbol{x} = \pi^{p/2}(\det A')^{-1/2}\exp\left[\boldsymbol{f}^T A'^{-1}\boldsymbol{f}/4\right]. \qquad (2.21)$$

With $N = p/2$, (2.20) can be obtained from (2.21) by the change of variables

$$\boldsymbol{x}^T = \left[\boldsymbol{\alpha}_1^T, \boldsymbol{\alpha}_2^T\right], \qquad (2.22)$$

$$A' = 2^{-1}\left[\begin{array}{c|c} A + A^T & j(A - A^T) \\ \hline -j(A - A^T) & A + A^T \end{array}\right], \qquad (2.23)$$

and

$$\boldsymbol{f}^T = 2j\left[\boldsymbol{\zeta}_2^T, -\boldsymbol{\zeta}_1^T\right], \qquad (2.24)$$

where $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$ and $\boldsymbol{\zeta}_1$, $\boldsymbol{\zeta}_2$ are the real and imaginary parts of $\boldsymbol{\alpha}$ and $\boldsymbol{\zeta}$, if A' has a positive definite real part. To complete the proof we will show that

$$A'' \equiv \left[\begin{array}{c|c} A_1 + A_1^T & A_2^T - A_2 \\ \hline A_2 - A_2^T & A_1 + A_1^T \end{array}\right] > \mathbf{0}, \qquad (2.25)$$

where A_1 and A_2 are the real and imaginary parts of A. The characteristic polynomials of A'', $A + A^\dagger$ and $(A + A^\dagger)^T$ are related by

$$\det\left[A'' - \lambda I\right] = \det\left[(A + A^\dagger) - \lambda I\right]\det\left[(A + A^\dagger)^T - \lambda I\right] \qquad (2.26)$$

which follows from the Szaraki–Wazewski relation [23, p. 85]. Equation (2.26) shows that (2.25) holds if and only if $A + A^\dagger$ is positive definite, so the proof is complete.

III. Operator Representation of Photodetection

As in [15], let $E(\bar{r},t)$, where $\bar{r}=(x,y,z)$, be the field operator[4] of a quasi-monochromatic scalar wave with nominal radian frequency ω_0 that illuminates a photoemissive surface in the $z=L$ plane which fills the circular pupil $A_L=\{\bar{x}\equiv(x,y)\colon |\bar{x}|\leq d_L/2\}$. Let $N(A\times\mathcal{T})$ denote the number of photoemissions that occur in an arbitrary time interval $\mathcal{T}\subseteq[t_0,\infty)$ within a spatial region $A\subseteq A_L$. We developed in [15] the statistics of the counting process $N(t)\equiv N(A_L\times[t_0,t))$ when the radiation field $E(\bar{r},t)$ is an arbitrary quantum state by building upon the photocounting theory of Kelley and Kleiner [7]. In this section we shall derive an operator representation for a photoemissive detector, a representation which facilitates the analysis of homodyne and heterodyne detection and offers further insight into the relationships between photoemissive detection and abstract quantum measurements. We begin by generalizing the sample-function density of [15] to include space–time counting processes.

Let $A_1,A_2,\cdots,A_p$ be a collection of arbitrary disjoint subsets of A_L, and let

$$N^T(t)=\left[N(A_1\times[t_0,t)),N(A_2\times[t_0,t)),\cdots,N(A_p\times[t_0,t))\right]$$

be the p-vector counting process associated with using A_1, $A_2,\cdots,A_p$ as a detector array. For any p the statistics of $N(t)$ can be obtained from those of the marked counting process $z(t)$ whose occurrence times are those of $N(t)$ and whose marks are the spatial locations of the counts [25, ch. 7]. As shown in Fig. 1, to completely specify a sample function $\{z(s)\colon t_0\leq s<t\}$ we need only know $\{(\bar{x},t_1)\colon 1\leq i\leq N(t)\}$, the space–time locations of the $N(t)$ counts that occur during $[t_0,t)$ over A_L. Thus, in parallel with [15, eq. (2.4)–(2.6)], we find the sample function density for $z(t)$ to be

$$p(\{z(s)\colon t_0\leq s<t\})=\begin{cases}\Pr[N(t)=0], & N(t)=0\\ p_{\bar{x},t}(\overline{X},\boldsymbol{T},N(t)=n), & N(t)=n\geq 1\end{cases}\tag{3.1}$$

where $(\bar{x},t)\equiv(\bar{x}_1,t_1;\bar{x}_2,t_2;\cdots;\bar{x}_n,t_n)$ are the ordered space–time occurrences,

$$\Pr[N(t)=0]=1+\sum_{m=1}^{\infty}(-1)^m\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2\cdots\int_{t_0}^{t}dt_m\cdot w_m(t_1,t_2,\cdots,t_m)/m!,\tag{3.2}$$

[4] In particular, the reader should be aware that our field operator notation suppresses a constant involving the photon energy at frequency ω_0 (see [13, eq. (2.15)]).

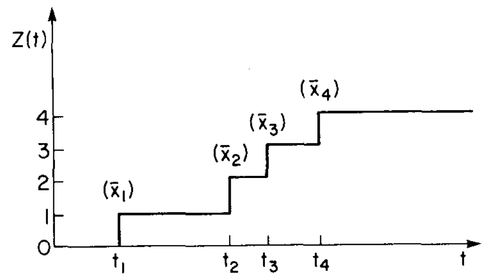

Fig. 1. Sample function of marked counting process $z(t)$.

is the probability that no counts occur during $[t_0,t]$, and

$$\begin{aligned}p_{\bar{x},t}(\overline{X},\boldsymbol{T},N(t)=n)&=\sum_{m=n}^{\infty}\left[(-1)^{m-n}/(m-n)!\right]\\&\cdot\int_{A_L}d\bar{x}_{n+1}\int_{t_0}^{t}dt_{n+1}\int_{A_L}d\bar{x}_{n+2}\\&\cdot\int_{t_0}^{t}dt_{n+2}\cdots\int_{A_L}d\bar{x}_m\int_{t_0}^{t}dt_m\\&\cdot w_m(\overline{X}_1,T_1;\overline{X}_2,T_2;\cdots;\overline{X}_n,T_n;\\&\bar{x}_{n+1},t_{n+1};\bar{x}_{n+2},t_{n+2};\cdots;\bar{x}_m,t_m)\end{aligned}\tag{3.3}$$

for $t_0<T_1<T_2<\cdots<T_n<t$ and $\overline{X}_i\in A_L$, $l\leq i\leq n$ is the joint probability density for the first n ordered occurrences to be at $(\overline{X}_1,T_1)$ through $(\overline{X}_n,T_n)$, respectively, and there be n counts during the interval $[t_0,t)$. In (3.2) and (3.3)

$$\begin{aligned}w_m(\bar{x}_1,t_1;\bar{x}_2,t_2;\cdots;\bar{x}_m,t_m)&\equiv\eta^m\mathrm{tr}(\rho E_L^\dagger(\bar{x}_1,t_1)E_L^\dagger(\bar{x}_2,t_2)\cdots E_L^\dagger(\bar{x}_m,t_m)\\&\cdot E_L(\bar{x}_1,t_1)E_L(\bar{x}_2,t_2)\cdots E_L(\bar{x}_m,t_m))\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}w_m(t_1,t_2,\cdots,t_m)&\equiv\int_{A_L}d\bar{x}_1\int_{A_L}d\bar{x}_2\cdots\int_{A_L}d\bar{x}_m\\&\cdot w_m(\bar{x}_1,t_1;\bar{x}_2,t_2;\cdots;\bar{x}_m,t_m)\end{aligned}\tag{3.5}$$

are multicoincidence rates [15], [19], [20], η is the quantum efficiency of the detector, and ρ is the density operator describing the state of the field $E_L(\bar{x},t)\equiv E(\bar{r},t)|_{z=L}$ over the space–time region of interest $A_L\times\mathcal{T}_L$, where $\mathcal{T}_L\equiv[t_0,t_0+T)$.

From the sample function density (3.1)–(3.3) the p-vector counting process statistics may be calculated via

$$N(t)=\int_0^t\int_{A_L}\boldsymbol{l}_A(\bar{x})\,dN_{\bar{x},t},\tag{3.6}$$

where $\boldsymbol{l}_A^T(\bar{x})=[l_{A_1}(\bar{x}),l_{A_2}(\bar{x}),\cdots,l_{A_p}(\bar{x})]$,

$$l_{A_i}(\bar{x})\equiv\begin{cases}1, & \bar{x}\in A_i\\ 0, & \bar{x}\notin A_i\end{cases}\tag{3.7}$$

are the indicator functions for the array, and

$$dN_{\bar{x},t} \equiv \lim_{\substack{\Delta t \to 0 \\ |\Delta \bar{x}| \to 0}} N([\bar{x}, \bar{x}+\Delta\bar{x}) \times [t, t+\Delta t))$$

is the space–time point process associated with $z(t)$. A much more convenient calculation for the statistics of $N(t)$ can be obtained from the following theorem, which gives an operator representation for $dN_{\bar{x},t}$.

Theorem 1: Let $z(t)$ be the marked counting process characterized by the sample function density (3.1)–(3.3). The characteristic functional ϕ_{dN} of $dN_{\bar{x},t}$, the point process associated with $z(t)$, satisfies

$$\phi_{dN}(jv) \equiv E\left[\exp\left(j\int_{A_L}\int_{t_0}^{t} v(\bar{x},s)\, dN_{\bar{x},s}\right)\right]$$
$$= \mathrm{tr}\left[\rho' \exp\left(j\int_{A_L} d\bar{x} \int_{t_0}^{t} ds\, v(\bar{x},s) \cdot E_L'^{\dagger}(\bar{x},s) E_L'(\bar{x},s)\right)\right] \quad (3.8)$$

for $t \in \mathfrak{T}_L$, where[5]

$$E_L'(\bar{x},t) = \eta^{1/2} E_L(\bar{x},t) + (1-\eta)^{1/2} E_v(\bar{x},t) \quad (3.9)$$

on $A_L \times \mathfrak{T}_L$. In (3.8) and (3.9), η is the detector quantum efficiency, E_v is a vacuum-state field, and ρ' is the density operator $\rho \otimes_n |0\rangle\langle 0|$ that jointly characterizes E_L and E_v on $A_L \times \mathfrak{T}_L$, where ρ is the density operator for E_L.

Proof: We begin by reducing ϕ_{dN} to a form that directly involves ρ. From the definition (3.8) we obtain

$$\phi_{dN}(jv) = \Pr[N(t)=0] + \sum_{n=1}^{\infty} \int_{A_L} d\bar{X}_n \int_{t_0}^{t} dT_n \exp\left[j\sum_{k=1}^{n} v(\bar{X}_k, T_k)\right] \cdot p_{x,t}(\bar{X}_n, T_n, N(t)=n). \quad (3.10)$$

By means of (3.2) and (3.3) and the invariance of the multicoincidence rates $w_m(\bar{x}_1, t_1; \bar{x}_2, t_2; \cdots; \bar{x}_m, t_m)$ to arbitrary permutations of the $(\bar{x}_i, t_i)$, we may rewrite (3.10) as follows:

$$\phi_{dN}(jv) = 1 + \sum_{m=1}^{\infty} (-1)^m \int_{t_0}^{t} dT_m w_m(T_1, T_2, \cdots, T_m)/m! + \sum_{n=1}^{\infty}\sum_{m=n}^{\infty} [(-1)^{m-n}/(m-n)!n!] \cdot \int_{A_L} d\bar{X}_m \int_{t_0}^{t} dT_m \exp\left[j\sum_{k=1}^{n} v(\bar{X}_k, T_k)\right] \cdot w_m(\bar{X}_1, T_1; \bar{X}_2, T_2; \cdots; \bar{X}_m, T_m). \quad (3.11)$$

[5] It is more legitimate to write (3.9) as $E_L'(\bar{x},t) = \eta^{1/2} E_L(\bar{x},t) \otimes \mathbf{1}' + (1-\eta)^{1/2} \mathbf{1} \otimes E_v(\bar{x},t)$ to emphasize that E_L and E_v operate on different Hilbert spaces and that E_L' is an operator obtained from E_L, E_v and the appropriate identity operators $\mathbf{1}'$ and $\mathbf{1}$ as shown. We shall continue to use the simpler notation (3.9), which is consistent with notation employed in similar contexts in [13] and [15]. As yet no physical locus for the field E_v has been identified.

Next, via (3.4) and (3.5), we may identify $\phi_{dN}(jv) = \mathrm{tr}(\rho M)$ where M is the operator

$$M = \sum_{n=0}^{\infty}\sum_{m=n}^{\infty} [\eta^m (-1)^{m-n}/(m-n)!n!] \cdot \int_{A_L} d\bar{X}_m \int_{t_0}^{t} dT_m \exp\left[j\sum_{k=1}^{n} v(\bar{X}_k, T_k)\right] \cdot E_L^{\dagger}(\bar{X}_1, T_1) E_L^{\dagger}(\bar{X}_2, T_2) \cdots E_L^{\dagger}(\bar{X}_m, T_m) \cdot E_L(\bar{X}_1, T_1) E_L(\bar{X}_2, T_2) \cdots E_L(\bar{X}_m, T_m). \quad (3.12)$$

To prove the theorem we shall reduce the right member of (3.8) to $\mathrm{tr}(\rho M)$. Because M defined by (3.12) is a normally ordered operator, our immediate task is to normal order the operator

$$\exp\left[j\int_{A_L} d\bar{x} \int_{t_0}^{t} ds\, v(\bar{x},s) E_L'^{\dagger}(\bar{x},s) E_L'(\bar{x},s)\right].$$

Let $\{\zeta_n(\bar{x},t)\}$ be a complete orthonormal (CON) set of functions on $A_L \times \mathfrak{T}_L$. In terms of $\{\zeta_n\}$ we may expand E_L, E_v, and E_L' in the series

$$E_L(\bar{x},t) = \sum_n a_n \zeta_n(\bar{x},t), \quad (3.13)$$

$$E_v(\bar{x},t) = \sum_n c_n \zeta_n(\bar{x},t), \quad (3.14)$$

and

$$E_L'(\bar{x},t) = \sum_n b_n \zeta_n(\bar{x},t) \quad (3.15)$$

on $A_L \times \mathfrak{T}_L$, where $\{a_n, c_n\}$ are photon annihilation operators which obey the CCR and

$$b_n \equiv \eta^{1/2} a_n + (1-\eta)^{1/2} c_n. \quad (3.16)$$

The state of the field modes associated with the $\{a_n\}$ is ρ; the field modes associated with the $\{c_n\}$ are all in the vacuum state. Using these modal expansions we can show that

$$M' \equiv \exp\left[j\int_{A_L} d\bar{x} \int_{t_0}^{t} ds\, v(\bar{x},s) E'^{\dagger}_L(\bar{x},s) E_L'(\bar{x},s)\right] = \exp[(\boldsymbol{d}^{\dagger})^T \boldsymbol{A} \boldsymbol{d}] \quad (3.17)$$

where $\boldsymbol{d}^T = [a_1, a_2, \cdots, a_n, \cdots, c_1, c_2, \cdots, c_n, \cdots]$,

$$\boldsymbol{A} = j\left[\begin{array}{c|c} \eta \boldsymbol{C} & \eta^{1/2}(1-\eta)^{1/2} \boldsymbol{C} \\ \hline \eta^{1/2}(1-\eta)^{1/2} \boldsymbol{C} & (1-\eta)\boldsymbol{C} \end{array}\right], \quad (3.18)$$

and $\boldsymbol{C}$ is the matrix whose mnth element is

$$(\boldsymbol{C})_{mn} = \int_{A_L} d\bar{x} \int_{t_0}^{t} ds\, v(\bar{x},s) \zeta_m^*(\bar{x},s) \zeta_n(\bar{x},s). \quad (3.19)$$

We shall use Lemma 1 to obtain the normally ordered form of M' even though $\boldsymbol{d}$ and $\boldsymbol{A}$ are infinite dimensional. (In practice there will be some finite number of space–time field modes which can influence the photodetector, hence by appropriately truncating the series (3.13)–(3.15) we can rigorously employ Lemma 1.) The result we obtain is

$$M'^{(n)}(\delta^*,\delta)=\exp[\delta^\dagger(e^A-I)\delta] \tag{3.20}$$

where $\delta^T=[\alpha_1,\alpha_2,\cdots,\alpha_n,\cdots,\gamma_1,\gamma_2,\cdots,\gamma_n,\cdots]$ gives the coherent amplutides associated with the $\{a_n,c_n\}$.

A differential equation method can be employed to show that

$$e^A\left[\begin{array}{c|c}\eta e^{jC}+(1-\eta)I & \eta^{1/2}(1-\eta)^{1/2}(e^{jC}-I)\\ \hline \eta^{1/2}(1-\eta)^{1/2}(e^{jC}-I) & (1-\eta)e^{jC}+\eta I\end{array}\right]. \tag{3.21}$$

We now find, using (2.6), that

$$\mathrm{tr}(\rho' M')=\mathrm{tr}(\rho\otimes_n|0\rangle\langle 0|M')=\mathrm{tr}(\rho M''), \tag{3.22}$$

where M'' is an operator whose normally ordered form is

$$M''^{(n)}(\alpha^*,\alpha)=\exp[\eta\alpha^\dagger(e^{jC}-I)\alpha]. \tag{3.23}$$

The last step of the proof is to show that $M''=M$; this is tedious to accomplish but straightforward as outlined below.

Using the series expansion for the matrix exponential (3.19) for $(C)_{mn}$ and the completeness relation

$$\sum_n \zeta_n^*(\bar{x},s)\zeta_n(\bar{x}',s')=\delta(\bar{x}-\bar{x}')\delta(s-s') \tag{3.24}$$

on $A_L\times\mathcal{T}_L$, it can be shown that

$$e^{jC}-I=\sum_{m=1}^{\infty} j^m C^m/m! \tag{3.25}$$

with

$$(C^m)_{ik}=\int_{A_L}d\bar{x}\int_{t_0}^{t}ds[v(\bar{x},s)]^m\xi_i^*(\bar{x},s)\xi_k(\bar{x},s)$$

which gives

$$(e^{jC}-I)_{ik}=\int_{A_L}d\bar{x}\int_{t_0}^{t}ds[\exp(jv(\bar{x},s))-1]\xi_i^*(\bar{x},s)\xi_k(\bar{x},s).$$

Thus we find that

$$\exp[\eta\alpha^\dagger(e^{jC}-I)\alpha]=\exp\left\{\eta\int_{A_L}d\bar{x}\int_{t_0}^{t}ds[\exp(jv(\bar{x},s))-1]\cdot\mathcal{E}_L^*(\bar{x},s)\mathcal{E}_L(\bar{x},s)\right\} \tag{3.27}$$

where $\mathcal{E}_L(\bar{x},s)\equiv\sum_i\alpha_i\xi_i(\bar{x},s)$ is the classical field associated with the coherent state $|\alpha\rangle$. The series expansion of (3.27) now yields

$$\begin{aligned}\exp[\eta\alpha^\dagger(e^{jC}-I)\alpha]&=\sum_{m=0}^{\infty}\left\{\eta\int_{A_L}d\bar{x}\int_{t_0}^{t}[\exp(jv(\bar{x},s))-1]\mathcal{E}_L^*(\bar{x},s)\mathcal{E}_L(\bar{x},s)\right\}^m/m!\\&=\sum_{m=0}^{\infty}[\eta^m/m!]\int_{A_L}d\bar{x}_m\int_{t_0}^{t}ds_m\left\{\prod_{i=1}^{m}[\exp(jv(\bar{x}_i,s_i))-1]\right\}\\&\quad\cdot\mathcal{E}_L^*(\bar{x}_1,s_1)\mathcal{E}_L^*(\bar{x}_2,s_2)\cdots\mathcal{E}_L^*(\bar{x}_m,s_m)\mathcal{E}_L(\bar{x}_1,s_1)\mathcal{E}_L(\bar{x}_2,s_2)\cdots\mathcal{E}_L(\bar{x}_m,s_m)\\&=\sum_{m=0}^{\infty}\sum_{n=0}^{m}[\eta^m(-1)^{m-n}/(m-n)!n!]\int_{A_L}d\bar{x}_m\int_{t_0}^{t}ds_m\exp\left[j\sum_{k=1}^{n}v(\bar{x}_k,s_k)\right]\\&\quad\cdot\mathcal{E}_L^*(\bar{x}_1,s_1)\mathcal{E}_L^*(\bar{x}_2,s_2)\cdots\mathcal{E}_L^*(\bar{x}_m,s_m)\mathcal{E}_L(\bar{x}_1,s_1)\mathcal{E}_L(\bar{x}_2,s_2)\cdots\mathcal{E}_L(\bar{x}_m,s_m)\end{aligned} \tag{3.28}$$

for the normally ordered form of M'', where the last equality makes use of the multinominal theorem. Interchanging the order of summation over m and n in (3.28) makes it transparent that (3.28) is also the normally ordered form of M (cf., (3.12) with $E_L\to\mathcal{E}_L$), hence $M''=M$ and the theorem is proved.

Remarks: Theorem 1 asserts that the statistics of the classical space-time point process $dN_{\bar{x},t}$ are the same as those obtained from the quantum measurement of $E_L'^\dagger(\bar{x},t)E_L'(\bar{x},t)d\bar{x}\,dt$. In particular, when $\eta=1$ we have $E_L'\equiv E_L$ so that photoemissive detector is a realization of the photon flux-density measurement $E_L^\dagger(\bar{x},t)\cdot E_L(\bar{x},t)d\bar{x}\,dt$. It has long been known that photoemissive detection of a single-mode radiation field with a unity quantum efficiency detector yields a measurement of the modal photon number operator, i.e., observation of $N(t_0+T)$ constitutes a measurement of $\mathfrak{N}=a^\dagger a$ for $\eta=1$, where a is the modal photon annihilation operator [7], [12]. Moreover, observation of $N(t_0+T)$ for a single-mode field when $\eta<1$ has been shown to realize the quantum measurement of $\mathfrak{N}'=b^\dagger b$ where $b=\eta^{1/2}a+(1-\eta)^{1/2}c$ for c a photon operator associated with a mode in the vacuum state [26]. Theorem 1 is thus the generalization of these results to space-time distributed observations of a multimode radiation field. To the best of our knowledge no such explicit result has previously appeared in the physics or communication literature.

To illustrate the power of Theorem 1 let us compute the mean vector and convariance matrix of the p-vector counting process $N(t)$. For a fully legitimate calculation we should use

$$v(\bar{x},s)\equiv\sum_{i=1}^{p}v_i l_{A_i}(\bar{x}) \tag{3.29}$$

to evaluate $m_N(t)\equiv E(N(t))$ from $\phi_{dN}(jv)$ via Theorem 1. It is easier, however, to proceed formally and use

$$E[dN_{\bar{x},t}]=\mathrm{tr}[\rho' E_L'^\dagger(\bar{x},t)E_L'(\bar{x},t)]d\bar{x}\,dt \tag{3.30}$$

to obtain

$$\begin{aligned}m_N(t)&=\mathrm{tr}\left[\rho'\int_{A_L}d\bar{x}\int_{t_0}^{t}ds\,l_A(\bar{x})E_L'^\dagger(\bar{x},s)E_L'(\bar{x},s)\right]\\&=\int_{A_L}d\bar{x}\int_{t_0}^{t}ds\,l_A(\bar{x})w_1(\bar{x},s),\end{aligned} \tag{3.31}$$

for $t \in \mathcal{T}_L$. The more legitimate approach yields the same result. Similarly, to calculate $\boldsymbol{K}_{NN}(t,u) \equiv E[\boldsymbol{N}(t)\boldsymbol{N}^T(u)] - \boldsymbol{m}_N(t)\boldsymbol{m}_N^T(u)$ for $t_0 \leqslant u \leqslant t$ we should use (3.8) with

$$v(\bar{x},s) \equiv \begin{cases} \sum_{i=1}^{p} (v_i + v_i') l_{A_i}(\bar{x}), & \text{for } t_0 \leqslant s < u \\ \sum_{i=1}^{p} (v_i l_{A_i}(\bar{x})), & \text{for } u \leqslant s < t. \end{cases} \tag{3.32}$$

The same final answer can be obtained using the formal approach, viz.,

$$\begin{aligned} E\left[dN_{\bar{x},t} dN_{\bar{x}',t'}\right] &= \mathrm{tr}\left[\rho' E_L'^{\dagger}(\bar{x},t) E_L'(\bar{x},t) E_L'^{\dagger} \right. \\ &\quad \left. \cdot (\bar{x}',t') E_L'(\bar{x}',t')\right] d\bar{x}\, d\bar{x}'\, dt\, dt' \\ &= \mathrm{tr}\left[\rho' E_L'^{\dagger}(\bar{x},t) E_L'(\bar{x},t)\right] \delta_{\bar{x}\bar{x}'} \delta_{tt'}\, d\bar{x}\, dt \\ &\quad + \mathrm{tr}\left[\rho' E_L'^{\dagger}(\bar{x},t) E_L'^{\dagger}(\bar{x}',t') E_L' \right. \\ &\quad \left. \cdot (\bar{x},t) E_L'(\bar{x}',t')\right] d\bar{x}\, d\bar{x}'\, dt\, dt' \end{aligned} \tag{3.33}$$

gives

$$\begin{aligned} E\left[\boldsymbol{N}(t)\boldsymbol{N}^T(u)\right] &= \mathrm{tr}\left[\rho' \int_{A_L} d\bar{x} \int_{t_0}^{\min(t,u)} ds\, \boldsymbol{l}_A(\bar{x}) \boldsymbol{l}_A^T(\bar{x}) E_L'^{\dagger} \right. \\ &\quad \left. \cdot (\bar{x},s) E_L'(\bar{x},s)\right] \\ &\quad + \mathrm{tr}\left[\rho' \int_{A_L} d\bar{x}_2 \int_{t_0}^{t} ds_1 \int_{t_0}^{u} ds_2\, \boldsymbol{l}_A(\bar{x}_1) \boldsymbol{l}_A^T(\bar{x}_2) \right. \\ &\quad \left. \cdot E_L'^{\dagger}(\bar{x}_1,s_1) E_L'^{\dagger}(\bar{x}_2,s_2) E_L'(\bar{x}_1,s_1) E_L'(\bar{x}_2,s_2)\right] \\ &= \int_{A_L} d\bar{x} \int_{t_0}^{\min(t,u)} ds\, \boldsymbol{l}_A(\bar{x}) \boldsymbol{l}_A^T(\bar{x}) w_1(\bar{x},s) \\ &\quad + \int_{A_L} d\bar{x} \int_{t_0}^{t} ds_1 \int_{t_0}^{u} ds_2\, \boldsymbol{l}_A(\bar{x}_1) \boldsymbol{l}_A^T(\bar{x}_2) \\ &\quad \cdot w_2(\bar{x}_1,s_1;\bar{x}_2,s_2), \end{aligned} \tag{3.34}$$

whence

$$\begin{aligned} \boldsymbol{K}_{NN}(t,u) &= \int_{A_L} d\bar{x} \int_{t_0}^{\min(t,u)} ds \\ &\quad \cdot \mathrm{diag}\left(l_{A_1}(\bar{x}), l_{A_2}(\bar{x}), \cdots, l_{A_p}(\bar{x})\right) \cdot w_1(\bar{x},s) \\ &\quad + \int_{A_L} d\bar{x}_2 \int_{t_0}^{t} ds_1 \int_{t_0}^{u} ds_2\, \boldsymbol{l}_A(\bar{x}_1) \boldsymbol{l}_A^T(\bar{x}_2) \\ &\quad \cdot \left[w_2(\bar{x}_1,s_1;\bar{x}_2,s_2) \right. \\ &\quad \left. - w_1(\bar{x}_1,s_1) w_1(\bar{x}_2,s_2)\right]. \end{aligned} \tag{3.35}$$

Note that (3.31) and (3.35) provide the p-vector generalization of Theorem 2 in [15], and (3.30) and (3.33), which may be rewritten as

$$E\left[dN_{\bar{x},t}\right] = w_1(\bar{x},t)\, d\bar{x}\, dt, \tag{3.36}$$

$$\begin{aligned} E\left[dN_{\bar{x},t} dN_{x',t'}\right] &= w_1(\bar{x},t) \delta_{\bar{x}\bar{x}'} \delta_{\bar{t}t'}\, d\bar{x}\, dt \\ &\quad + w_2(\bar{x},t;\bar{x}',t')\, d\bar{x}\, d\bar{x}'\, dt\, dt', \end{aligned} \tag{3.37}$$

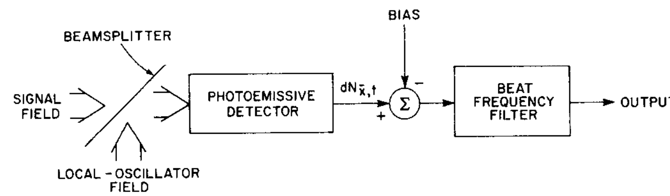

Fig. 2. Configuration for homodyne and heterodyne detection: for homodyne detection the beat frequency filter is a baseband filter; for heterodyne detection the bias subtraction is redundant, and the beat frequency filter is an intermediate frequency (IF) filter.

are space–time versions of the formal results given in [15] for $dN_t = \lim_{\Delta t \to 0}[N(t+\Delta t) - N(t)]$.

IV. Homodyne Detection and Single-Quadrature Field Measurement

In homodyne detection of a quasi-monochromatic signal field $E_s(\bar{x},t)$ of nominal frequency ω_0, a beamsplitter is used to mix $E_s(\bar{x},t)$ with $E_l(\bar{x},t)$, the field of a strong local-oscillator laser also of frequency ω_0, on the active region of a photodetector as shown in Fig. 2. The point process output of the photodetector is then filtered to extract its baseband beat frequency components and a fixed bias level (which is due to the local-oscillator laser) is removed. We wish to show that under appropriate conditions this homodyne detection scheme realizes the quantum field-quadrature measurement, i.e., it measures

$$2^{-1}\left[E_s(\bar{x},t)e^{j\omega_0 t} + E_s^{\dagger}(\bar{x},t)e^{-j\omega_0 t}\right]$$

for *arbitrary* states of the multimode radiation field. We begin our investigation with some simple second-moment analysis.

The beamsplitter in Fig. 2 will be taken to be lossless so that $E_d(\bar{x},t)$, the field incident on the photodetector, satisfies

$$E_d(\bar{x},t) = \epsilon^{1/2} E_s(\bar{x},t) + (1-\epsilon)^{1/2} E_l(\bar{x},t), \tag{4.1}$$

where ϵ is the beamsplitter's intensity transmission coefficient. As in [15], we know that the density operator $\rho = \rho_s \otimes \rho_l$ jointly characterizes E_s and E_l on $A_L \times \mathcal{T}_L$, where ρ_s is the state of E_s and ρ_l is the state of E_l. Ordinarily, the local-oscillator field will have a single nonvacuum state mode which is in a coherent state $|N_l^{1/2}\rangle$ with $N_l \gg 1$. Using the expansions

$$E_s(\bar{x},t) = \sum_n a_n \xi_n(\bar{x},t) \tag{4.2}$$

and

$$E_l(\bar{x},t) = \sum_n c_n \xi_n(\bar{x},t) \tag{4.3}$$

where $\{\xi_n\}$ is a CON set on $A_L \times \mathcal{T}_L$, and ξ_1 is the nonvacuum state mode of E_l, the mean and covariance functions of $dN_{\bar{x},t}$ are easily derived from (3.36) and (3.37) as discussed in [15, sec. III]. In particular, for the point process[6]

$$dN'_{\bar{x},t}=\lim_{N_l\to\infty}\left\{\left[4\eta^2N_l\epsilon(1-\epsilon)|\xi_1(\bar{x},t)|^2\right]^{-1/2}\cdot\left[dN_{\bar{x},t}-\eta(1-\epsilon)N_l|\xi_1(\bar{x},t)|^2dx\,dt\right]\right\}, \quad (4.4)$$

we find that

$$E\left[dN'_{\bar{x},t}\right]=\mathrm{tr}\left[\rho_sF(\bar{x},t)\right]d\bar{x}\,dt \quad (4.5)$$

and

$$E\left[dN'_{\bar{x},t}dN'_{\bar{x}',t'}\right]-E\left[dN'_{\bar{x},t}\right]E\left[dN'_{\bar{x}',t'}\right]=\mathrm{tr}\left[\rho_s\Delta F(\bar{x},t)\Delta F(\bar{x}',t')\right]d\bar{x}\,d\bar{x}'\,dt\,dt'+\left[(1-\eta\epsilon)/4\eta\epsilon\right]\delta_{\bar{x}\bar{x}'}\delta_{tt'}\,d\bar{x}\,dt, \quad (4.6)$$

where

$$F(\bar{x},t)\equiv2^{-1}\left[E_s(\bar{x},t)e^{-j\phi(\bar{x},t)}+E_s^\dagger(\bar{x},t)e^{j\phi(\bar{x},t)}\right], \quad (4.7)$$

$$\Delta F\equiv F-\mathrm{tr}\left[\rho_sF\right], \quad (4.8)$$

and $\phi(\bar{x},t)\equiv\arg[\xi_1(\bar{x},t)]$.

Equations (4.4)–(4.8) are *consistent with* the assertion that in the lossless limit $\eta\epsilon\to1$ observation of $dN'_{\bar{x},t}$ constitutes a measurement of $F(\bar{x},t)$; note that in the typical case of a monochromatic plane-wave local oscillator we will have $\phi(\bar{x},t)=-\omega_0t$ so that $F(\bar{x},t)=2^{-1}[E_s(\bar{x},t)e^{j\omega_0t}+E_s^\dagger(\bar{x},t)e^{-j\omega_0t}]$. That observation of $dN'_{\bar{x},t}$ in the limit $\eta\epsilon\to1$ does realize the $F(\bar{x},t)$ measurement is shown in the following theorem.

Theorem 2: Let $dN'_{\bar{x},t}$ be the point process generated via homodyne detection according to (4.4). The characteristic functional of $dN'_{\bar{x},t}$ obeys

$$\phi_{dN'}jv\equiv E\left[\exp\left(j\int_{A_L}\int_{\mathcal{T}_L}v(\bar{x},t)\,dN'_{\bar{x},t}\right)\right]=\exp\left[-(1-\eta\epsilon)\int_{A_L}d\bar{x}\int_{\mathcal{T}_L}dt\,v^2(\bar{x},t)/8\eta\epsilon\right]\cdot\mathrm{tr}\left[\rho_s\exp\left(j\int_{A_L}d\bar{x}\int_{\mathcal{T}_L}dt\,v(\bar{x},t)F(\bar{x},t)\right)\right], \quad (4.9)$$

where $F(\bar{x},t)$ is given by (4.7).

Proof: We have that

$$E\left[\exp\left(j\int_{A_L}\int_{\mathcal{T}_L}v(\bar{x},t)\left[4\eta^2N_l\epsilon(1-\epsilon)|\xi_1(\bar{x},t)|^2\right]^{-1/2}\cdot\left[dN_{\bar{x},t}-\eta(1-\epsilon)N_l|\xi_1(\bar{x},t)|^2d\bar{x}\,dt\right]\right)\right]=\exp\left\{-j\left[\eta(1-\epsilon)N_l/K\right]\cdot\int_{A_L}d\bar{x}\int_{\mathcal{T}_L}dt\,v'(\bar{x},t)|\xi_1(\bar{x},t)|^2\right\}\cdot\phi_{dN}(jv'/K) \quad (4.10)$$

with $v'(\bar{x},t)\equiv v(\bar{x},t)/|\xi_1(\bar{x},t)|$, and $K\equiv[4\eta^2(1-\epsilon)\epsilon N_l]^{1/2}$. From Theorem 1 we know that

$$\phi_{dN}(jv)=\mathrm{tr}\left[\rho'\exp\left(j\int_{A_L}d\bar{x}\int_{\mathcal{T}_L}dt\,v(\bar{x},t)\cdot E_d'^\dagger(\bar{x},t)E_d'(\bar{x},t)\right)\right], \quad (4.11)$$

for $E_d'(\bar{x},t)=\eta^{1/2}E_d(\bar{x},t)+(1-\eta)^{1/2}E_v(\bar{x},t)$ and $\rho'=\rho_b\otimes_n|0\rangle\langle0|$ a density operator which jointly characterizes E_d and E_v. Following the line of reasoning employed in (3.13)–(3.23), we can show[7] that the operator

$$M_K=\exp\left(j\int_{A_L}d\bar{x}\int_{\mathcal{T}_L}dt\,v'(\bar{x},t)E_d'^\dagger(\bar{x},t)E_d'(\bar{x},t)/K\right) \quad (4.12)$$

has the normally ordered form

$$M_K^{(n)}(\alpha^*,\alpha)=\exp\left[\eta\alpha^\dagger(\exp(jC'/K)-I)\alpha\right], \quad (4.13)$$

where

$$(C')_{mn}=\int_{A_L}d\bar{x}\int_{\mathcal{T}_L}dt\,v'(\bar{x},t)\xi^*_m(\bar{x},t)\xi_n(\bar{x},t).$$

Because C' Hermitian, the matrix

$$\left[I-\exp(jC'/K)\right]+\left[I-\exp(jC/K)\right]^\dagger=2\left[I-\cos(C'/K)\right]$$

is positive definite for K sufficiently large but still finite. Thus via Lemma 2 the Fourier transform of (4.13) is

$$\chi_A^{M_K}(\zeta^*,\zeta)=\left\{\det\left[\eta(I-\exp(jC'/K))\right]\right\}^{-1}\cdot\exp\left\{-\zeta^\dagger\left[\eta(I-\exp(jC'/K))\right]^{-1}\zeta\right\}, \quad (4.14)$$

and by means of (2.5) and (2.6b) we find that

$$\phi_{dN}(jv'/K)=\left\{\det\left[\eta(I-\exp(jC'/K))\right]\right\}^{-1}\cdot\int\exp\left\{-\zeta^\dagger\left[\eta^{-1}(I-\exp(jC'/K))^{-1}-I/2\right]\zeta\right\}\cdot\chi_W^{\rho_b}(\zeta^*,\zeta)d^2\zeta/\pi^N. \quad (4.15)$$

In (4.15), ρ_b is the density operator for the field E_d, and N is some finite number of modes which is sufficient for characterizing the post-detection filtered point process; i.e., we can assume that the series

$$E_d(\bar{x},t)=\sum_n\left[\epsilon^{1/2}a_n+(1-\epsilon)^{1/2}c_n\right]\xi_n(\bar{x},t)\equiv\sum b_n\xi_n(\bar{x},t) \quad (4.16)$$

is truncated to N terms.

[6] We assume henceforth that $|\xi_1(\bar{x},t)|>0$ on $A_L\times\mathcal{T}_L$.

[7] As in Section III, this derivation is couched in terms of infinite mode sets. However, if $dN_{\bar{x},t}$ is observed through a post-detection filter which has finite spatio-temporal bandwidth, only a finite number of field modes contribute to the filter's output, hence Lemmas 1 and 2 can be employed.

Now, as in [13, sec. II], we can show that

$$\chi_W^{\rho_l}(\zeta^*,\zeta)=\chi_W^{\rho_s}(\epsilon^{1/2}\zeta^*,\epsilon^{1/2}\zeta)\cdot\chi_W^{\rho_l}((1-\epsilon)^{1/2}\zeta^*,(1-\epsilon)^{1/2}\zeta)=\chi_W^{\rho_s}(\epsilon^{1/2}\zeta^*,\epsilon^{1/2}\zeta)\cdot\exp\{-(1-\epsilon)\zeta^\dagger\zeta/2+[(1-\epsilon)N_l]^{1/2}(\zeta-\zeta^*)^T u\}, \quad (4.17)$$

with $u^T\equiv[1,\,0,\,0,\cdots,\,0]$. This gives us, via a change of variables, that

$$\phi_{dN}(jv'/K)=\{\det[\eta\epsilon(I-\exp(jC'/K))]\}^{-1}\cdot\int\chi_W^{\rho_s}(\zeta^*,\zeta)\exp\{-\zeta^\dagger D\zeta+[(1-\epsilon)N_l/\epsilon]^{1/2}(\zeta-\zeta^*)^T u\}d^2\zeta/\pi^N, \quad (4.18)$$

$$D\equiv[\eta\epsilon(I-\exp(jC'/K))]^{-1}-I/2. \quad (4.19)$$

Our final task is to take the limit $N_l\to\infty$. Substituting the inverse relation of (2.4) into (4.18) we get

$$\phi_{dN}(jv'/K)=\{\det[\eta\epsilon(I-\exp(jC'/K))]\}^{-1}\cdot\int W^{\rho_s}(\alpha^*,\alpha)\pi^{-N}d^2\alpha\cdot\int\exp\{-\zeta^\dagger D\zeta+\zeta^T(\alpha^*+[(1-\epsilon)N_l/\epsilon]^{1/2}u)-\zeta^\dagger(\alpha+[(1-\epsilon)N_l/\epsilon]^{1/2}u)\}d^2\zeta/\pi^N. \quad (4.20)$$

The inner integral in the above expression can be evaluated by means of Lemma 2; to show that $D+D^\dagger>0$ we note that this matrix is diagonalized by the unitary transformation which diagonalizes C' and that

$$[\eta\epsilon(1-e^{jv/K})]^{-1}+[\eta\epsilon(1-e^{-jv/K})]^{-1}-1=[1-\eta\epsilon]/\eta\epsilon>0$$

for $0<\eta\epsilon<1$ and real v. Thus (4.20) becomes

$$\phi_{dN}(jv'/K)=\{\det[I-\eta\epsilon(I-\exp(jC'/K))/2]\}^{-1}\cdot\int W^{\rho_s}(\alpha^*,\alpha)\exp[-\beta^\dagger D^{-1}\beta]d^2\alpha/\pi^N, \quad (4.21)$$

with $\beta\equiv\alpha+[(1-\epsilon)N_l/\epsilon]^{1/2}u$. As $N_l\to\infty$ we have $K\to\infty$, and the determinant term in (4.21) goes to unity. Also, as $K\to\infty$ we can show that

$$D^{-1}=\eta\epsilon[-jC'/K+(1-\eta\epsilon)C'^2/2K^2+\cdots]; \quad (4.22)$$

this result follows from using the Laurent series

$$[\eta\epsilon(1-e^{jv/K})]^{-1}-\tfrac{1}{2}=\eta\epsilon[-jv/K+v^2(1-\eta\epsilon)/2K^2+\cdots]$$

to expand D^{-1} via the diagonal representation of C'. Substitution of (4.22) into (4.21) enables us to compute for $N_l\to\infty$ that

$$\phi_{dN}(jv'/K)=\int W^{\rho_s}(\alpha^*,\alpha)\cdot\exp\{j[\eta(1-\epsilon)N_l/K]u^TC'u+j[\alpha^\dagger C'u+u^TC'\alpha]/2-(1-\eta\epsilon)u^TC'^2u/8\eta\epsilon\}d^2\alpha/\pi^N$$
$$=\int W^{\rho_s}(\alpha^*,\alpha)\exp\Big\{j[\eta(1-\epsilon)N_l/K]\cdot\int_{A_L}d\bar{x}\int_{\mathcal{T}_L}dt\,v'(\bar{x},t)|\xi_1(\bar{x},t)|^2+j\int_{A_L}d\bar{x}\int_{\mathcal{T}_L}dt\,v(\bar{x},t)\mathcal{F}(\bar{x},t)-(1-\eta\epsilon)\cdot\int_{A_L}d\bar{x}\int_{\mathcal{T}_L}dt\,v^2(\bar{x},t)/8\eta\epsilon\Big\}d^2\alpha/\pi^N, \quad (4.23)$$

where $\mathcal{F}(\bar{x},t)$ is the classical field that (4.7) associates with the coherent state $|\alpha\rangle$. At last, by substituting (4.23) into (4.10), we see that

$$\phi_{dN'}(jv)=\int W^{\rho_s}(\alpha^*,\alpha)\cdot\exp\Big\{j\int_{A_L}d\bar{x}\int_{\mathcal{T}_L}dtv(\bar{x},t)\mathcal{F}(\bar{x},t)-(1-\eta\epsilon)\int_{A_L}d\bar{x}\int_{\mathcal{T}_L}dtv^2(\bar{x},t)/8\eta\epsilon\Big\}d^2\alpha/\pi^N$$
$$=\exp\Big[-(1-\eta\epsilon)\int_{A_L}d\bar{x}\int_{\mathcal{T}_L}dtv^2(\bar{x},t)/8\eta\epsilon\Big]\cdot\mathrm{tr}\Big[\rho_s\exp\Big(j\int_{A_L}d\bar{x}\int_{\mathcal{T}_L}dtv(\bar{x},t)F(\bar{x},t)\Big)\Big], \quad (4.24)$$

which proves the theorem.

Remarks: Theorem 2 tells us that observation of $dN'_{\bar{x},t}$ is equivalent to measuring. $F(\bar{x},t)d\bar{x}\,dt$ in the presence of a classical space–time white Guassian noise of spectral height $(1-\eta\epsilon)/4\eta\epsilon$; in the lossless limit $\eta\epsilon\to1$, homodyne detection therefore realizes the field-quadrature quantum measurement. For example, if $\xi_1(\bar{x},t)=(4/\pi d_L^2T)^{1/2}e^{-j\omega_0 t}$ and the ξ_1 mode of E_s is in the TCS $|\mu\nu\alpha\rangle$, then the random variable

$$y\equiv\int_{A_L}\int_{\mathcal{T}_L}(4/\pi d_L^2T)^{1/2}dN'_{\bar{x},t}=\lim_{N_l\to\infty}\{[N(t_0+T)-\eta(1-\epsilon)N_l]\cdot[4\eta^2\epsilon(1-\epsilon)N_l]^{-1/2}\} \quad (4.25)$$

is Guassian with mean value

$$E(y)=\mathrm{tr}[\rho_s(a_s+a_s^\dagger)/2]=\mathrm{Re}(\alpha) \quad (4.26)$$

and variance

$$\mathrm{Var}(y)=\mathrm{tr}\left[\rho_s(\Delta a_{s1})^2\right]+(1-\eta\epsilon)/4\eta\epsilon$$
$$=|\mu-\nu|^2/4+(1-\eta\epsilon)/4\eta\epsilon \qquad (4.27)$$

where a_s is the photon operator associated with the ξ_1 mode of E_s, and $a_{s1}\equiv(a_s+a_s^\dagger)/2$. For $\mu,\nu>0$, the TCS noise contribution to (4.27) is *less* than the corresponding coherent state value of $\frac{1}{4}$, and when $(1-\eta\epsilon)/\eta\epsilon \ll |\mu-\nu|^2$ the homodyne system behaves as though it were lossless.

Two final points concerning homodyne detection are worthy of mention. First, it is not necessary to use a multimode local oscillator to observe a multimode signal field. Indeed, if $dN'_{\bar{x},t}$ can be obtained, all the space–time models of E_s are available and individual modes can be selected by spatio-temporal filtering of $dN'_{\bar{x},t}$ as in (4.25). Second, the noise observed in homodyne detection when $\eta\epsilon\to 1$ is *entirely* due to signal-field fluctuations. In particular, the observed fluctuations are *not* due to local-oscillator shot noise as predicted from semiclassical photodetection theory [27].

V. TCS-Heterodyne Detection and Canonical Measurements

In heterodyne detection a signal field is mixed through a beamsplitter with a local oscillator field, whose nominal frequency is offset by an amount ω_{IF} from that of the signal, on the surface of a photodetector as shown in Fig. 2. The point process output of the photodetector is then filtered to select its beat frequency components in the vicinity of the intermediate frequency (IF) ω_{IF}. We shall develop in this section the quantum statistics of heterodyne detection as a special case of a more general configuration, defined below, which we will term TCS-heterodyne detection.

As in Section IV, let the beamsplitter in Fig. 2 be lossless, so that $E_d(\bar{x},t)$, the field incident on the photodetector, satisfies

$$E_d(\bar{x},t)=\epsilon^{1/2}E(\bar{x},t)+(1-\epsilon)^{1/2}E_l(\bar{x},t) \qquad (5.1)$$

in terms of the field $E(x,t)$ entering the beamsplitter from the left, the local-oscillator field $E_l(\bar{x},t)$, and the beamsplitter's intensity transmission coefficient ϵ. The local-oscillator field will be assumed to have a single non-vacuum state mode of nominal frequency ω_0; i.e., as in Section IV we use the expansion

$$E_l(\bar{x},t)=\sum_n c_n\xi_n(\bar{x},t) \qquad (5.2)$$

on $A_L\times\mathfrak{T}_L$ where $\{\xi_n\}$ is a CON set, the ξ_1 mode is of nominal frequency ω_0, and in the coherent state $|N_l^{1/2}\rangle$ with $N_l\gg 1$ and all other modes are in the vacuum state.

It follows immediately from Theorem 2 that the vector point process defined by

$$dN'^{T}_{\bar{x},t}=\left[{}_1dN'_{\bar{x},t},\,{}_2dN'_{\bar{x},t}\right], \qquad (5.3)$$
$${}_1dN'_{\bar{x},t}=2\cos(\omega_{\mathrm{IF}}t)\,dN'_{\bar{x},t}, \qquad (5.4)$$
$${}_2dN'_{\bar{x},t}=2\sin(\omega_{\mathrm{IF}}t)\,dN'_{\bar{x},t}, \qquad (5.5)$$

with $dN'_{\bar{x},t}$ given by (4.4), has the characteristic functional

$$\phi_{dN'}(jv)\equiv E\left[\exp\left(j\int_{A_L}\int_{\mathfrak{T}_L}v^T(\bar{x},t)\,dN'_{\bar{x},t}\right)\right]$$
$$=\exp\left[-(1-\eta\epsilon)\int_{A_L}d\bar{x}\int_{\mathfrak{T}_L}dt\left[v_1(\bar{x},t)\cos(\omega_{\mathrm{IF}}t)+v_2(\bar{x},t)\sin(\omega_{\mathrm{IF}}t)\right]^2/2\eta\epsilon\right]$$
$$\cdot\mathrm{tr}\left[\rho\exp\left(j\int_{A_L}d\bar{x}\int_{\mathfrak{T}_L}dt\,v^T(\bar{x},t)F(\bar{x},t)\right)\right] \qquad (5.6)$$

where

$$F^T(\bar{x},t)=\left[F_1(\bar{x},t),F_2(\bar{x},t)\right], \qquad (5.7)$$
$$F_1(\bar{x},t)=\left[E(\bar{x},t)e^{-j\phi(\bar{x},t)}+E^\dagger(\bar{x},t)e^{j\phi(\bar{x},t)}\right]\cos(\omega_{\mathrm{IF}}t), \qquad (5.8)$$
$$F_2(\bar{x},t)=\left[E(\bar{x},t)e^{-j\phi(\bar{x},t)}+E^\dagger(\bar{x},t)e^{j\phi(\bar{x},t)}\right]\sin(\omega_{\mathrm{IF}}t), \qquad (5.9)$$

ρ is the density operator for $E(\bar{x},t)$, and $\phi(\bar{x},t)=\arg[\xi_1(\bar{x},t)]$. Now, because post-detection filtering will be used to select the frequency components of $dN_{\bar{x},t}$ that lie within some finite bandwidth $\Delta\omega$ about ω_{IF}, we may, assuming $\Delta\omega\ll\omega_{\mathrm{IF}}$, omit the bias-subtraction in calculating $dN'_{\bar{x},t}$. Furthermore, when $\Delta\omega\ll\omega_{\mathrm{IF}}$, and the signal field has nominal frequency $\omega_0+\omega_{\mathrm{IF}}$, we may use

$$E(\bar{x},t)=E_s(\bar{x},t)+E_i(\bar{x},t) \qquad (5.10)$$

in (5.8) and (5.9), where

$$E_s(\bar{x},t)\equiv{\sum_n}'\,a_n\xi_n(\bar{x},t)e^{-j\omega_{\mathrm{IF}}t} \qquad (5.11)$$

and

$$E_i(\bar{x},t)\equiv{\sum_n}'\,e_n\xi_n(\bar{x},t)e^{j\omega_{\mathrm{IF}}t} \qquad (5.12)$$

are, respectively, the signal and image fields,[8] and $\sum_n'$ is a truncated summation which includes index n only if the mode $\xi_n(\bar{x},t)$ has appreciable spectral content within $\Delta\omega$ of ω_0. Thus by eliminating double-frequency terms we can reduce (5.6) to

$$\phi_{dN'}(jv)=\exp\left[-(1-\eta\epsilon)\int_{A_L}d\bar{x}\int_{\mathfrak{T}_L}dt\,v^T(\bar{x},t)v(\bar{x},t)/4\eta\epsilon\right]$$
$$\cdot\mathrm{tr}\left[\rho_s\otimes\rho_i\exp\left(j\int_{A_L}d\bar{x}\int_{\mathfrak{T}_L}dt\,v^T(\bar{x},t)F'(\bar{x},t)\right)\right], \qquad (5.13)$$

[8] Physically, $E_i(\bar{x},t)$, which is a field with nominal frequency $\omega_0-\omega_{\mathrm{IF}}$, describes the image-band components of $E(\bar{x},t)$ that, in addition to the signal-band field $E_s(\bar{x},t)$, beat with $E_l(\bar{x},t)$ to produce the observed IF output. The importance of the image field in heterodyne detection has previously been pointed out by Personick [28].

where

$$F_1'(\bar{x},t)=2^{-1}\left[E_s(\bar{x},t)e^{j(\omega_{\mathrm{IF}}t-\phi(\bar{x},t))}+E_s^\dagger(\bar{x},t)e^{-j(\omega_{\mathrm{IF}}t-\phi(\bar{x},t))}\right]+2^{-1}\left[E_i(\bar{x},T)e^{-j(\omega_{\mathrm{IF}}t+\phi(\bar{x},t))}+E_i^\dagger(\bar{x},t)e^{j(\omega_{\mathrm{IF}}t+\phi(\bar{x},t))}\right] \quad (5.14)$$

$$F_2'(\bar{x},t)=(2j)^{-1}\left[E_s(\bar{x},t)e^{j(\omega_{\mathrm{IF}}t-\phi(\bar{x},t))}-E_s^\dagger(\bar{x},t)e^{-j(\omega_{\mathrm{IF}}t-\phi(\bar{x},t))}\right]-(2j)^{-1}\left[E_i(\bar{x},t)e^{-j(\omega_{\mathrm{IF}}t+\phi(\bar{x},t))}-E_i^\dagger(\bar{x},t)e^{j(\omega_{\mathrm{IF}}t+\phi(\bar{x},t))}\right], \quad (5.15)$$

and ρ_s, ρ_i are the density operators for E_s, E_i, which are assumed to be independent fields.

Equations (5.13)–(5.15) show that observation of the baseband components of $dN'_{\bar{x},t}$ (which are equivalent to the IF components of $dN_{\bar{x},t}$) correspond to an abstract quantum measurement of $F'(\bar{x},t)d\bar{x}\,dt$ embedded in an independent additive classical space–time white circulo-complex Guassian noise of spectral height $(1-\eta\epsilon)/\eta\epsilon$. In the lossless limit $\eta\epsilon\to 1$, $dN'_{\bar{x},t}$ realizes the $F(\bar{x},t)$ measurement. This is our fundamental result for heterodyne detection. Note that when $\eta\epsilon\to 1$ the statistics of $dN'_{\bar{x},t}$ are determined entirely by the signal and image field states. It only remains for us to explicitly determine the effect of the image field state; this is easily accomplished using quantum characteristic functions, viz.,

$$\begin{aligned}\mathrm{tr}\left[\rho_s\otimes\rho_i\exp\left(j\int_{A_L}d\bar{x}\int_{\mathfrak{T}_L}dt\,v^T(\bar{x},t)F'(\bar{x},t)\right)\right]&=\chi_W^{\rho_s}(\zeta^*,\zeta)\chi_W^{\rho_i}(\zeta,\zeta^*)\\&=\exp(|\zeta|^2)\chi_A^{\rho_s}(\zeta^*,\zeta)\chi_A^{\rho_i}(\zeta,\zeta^*),\end{aligned} \quad (5.16)$$

with

$$\zeta_n\equiv 2^{-1}\int_{A_L}d\bar{x}\int_{\mathfrak{T}_L}dt\,\xi_n^*(\bar{x},t)\left[jv_1(\bar{x},t)-v_2(\bar{x},t)\right]e^{j\phi(\bar{x},t)}.$$

In conventional heterodyne detection the image field will be in the vacuum state, hence $\chi_A^{\rho_i}(\zeta,\zeta^*)=\exp(-|\zeta|^2)$. More generally, because E_s and E_i occupy nonoverlapping frequency bands, we can in principle use a beamsplitter arrangement to place E_i in some desired state without affecting E_s. Thus, let us consider the TCS-heterodyne detection system, shown in Fig. 3, in which E_i is placed in the multimode TCS $|\mu^*\nu^*\mathbf{0}\rangle\equiv\otimes_n|\mu^*\nu^*0\rangle$ by means of an image-band oscillator[9] and a beamsplitter. For $\rho_i=|\mu^*\nu^*\mathbf{0}\rangle\langle\mu^*\nu^*\mathbf{0}|$ we have from (2.14)

$$\chi_A^{\rho_i}(\zeta,\zeta^*)=\exp\left[-|\mu|^2|\zeta|^2-\mathrm{Re}(\zeta^\dagger\zeta^*\mu^*\nu)\right].$$

(Note that the TCS $|\mu^*\nu^*\mathbf{0}\rangle$ becomes the vacuum state

[9] Although there exist generation schemes for producing the state $|\mu^*\nu^*\mathbf{0}\rangle$, it is probably more practical to generate $|\mu^*\nu^*\boldsymbol{\alpha}\rangle$ for $\boldsymbol{\alpha}\neq\mathbf{0}$ [14]. However, the use of $|\mu^*\nu^*\boldsymbol{\alpha}\rangle$ for the image-band oscillator state merely changes the mean values of the observations F_1', F_2' in a known way. Thus, we opt for the simpler calculation using $|\mu^*\nu^*\mathbf{0}\rangle$.

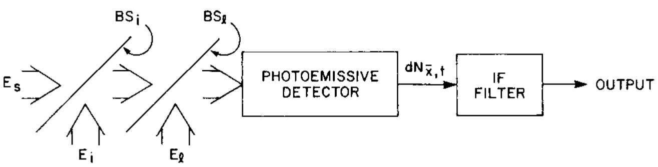

Fig. 3. Configuration for TCS-heterodyne detection: E_s is the signal field to be measured, E_i is the TCS field produced by the image-band oscillator, and E_l is the local-oscillator field; image-oscillator beamsplitter BS_i has near unity transmission coefficient for signal-band frequencies and near unity reflection for image-band frequencies; BS_l is local-oscillator beamsplitter.

$|\mathbf{0}\rangle$ when $\mu=1$, $\nu=0$, so that TCS-heterodyne detection includes the conventional case.) TCS-heterodyne detection therefore leads to the characteristic functional

$$\begin{aligned}\phi_{dN'}(jv)&=\exp\left[-(1-\eta\epsilon)\cdot\int_{A_L}d\bar{x}\int_{\mathfrak{T}_L}dt\,v^T(\bar{x},t)v(\bar{x},t)/4\eta\epsilon\right]\cdot\chi_A^{\rho_s}(\zeta^*,\zeta)\exp\left[-|\nu|^2|\zeta|^2-\mathrm{Re}(\zeta^\dagger\zeta^*\mu^*\nu)\right]\\&=\exp\left[-(1-\eta\epsilon)\int_{A_L}d\bar{x}\int_{\mathfrak{T}_L}dt\,v^T(\bar{x},t)\cdot v(\bar{x},t)/4\eta\epsilon\right]\chi_T^{\rho_s}(\zeta^*,\zeta),\end{aligned} \quad (5.17)$$

where $\chi_T^{\rho_s}$ is given by (2.17).

Equation (5.17) implies that TCS-heterodyne detection, in the limit $\eta\epsilon\to 1$, realizes the quantum measurement described by the POM $|\mu\nu\boldsymbol{\alpha}\rangle\langle\mu\nu\boldsymbol{\alpha}|$, i.e., $\langle\mu\nu\boldsymbol{\alpha}|\rho_s|\mu\nu\boldsymbol{\alpha}\rangle/\pi^N$ is the classical probability density for observing

$$dN'^T_{\bar{x},t}=\left[\mathrm{Re}\left(\mathcal{E}_s(\bar{x},t)e^{j(\omega_{\mathrm{IF}}t-\phi(\bar{x},t))}\right),\ \mathrm{Im}\left(\mathcal{E}_s(\bar{x},t)e^{j(\omega_{\mathrm{IF}}t-\phi(\bar{x},t))}\right)\right]d\bar{x}\,dt, \quad (5.18)$$

where $\mathcal{E}_s(\bar{x},t)\equiv\Sigma_n'\alpha_n\xi_n(\bar{x},t)e^{-j\omega_{\mathrm{IF}}t}$, and N is the number of modes included in Σ_n'. By allowing the image-band oscillator to produce a field in an arbitrary multimode TCS embedded in classical additive Guassian noise, it can be shown that the configuration in Fig. 3 realizes the canonical measurements described by Holevo [3], [16] when $\eta\epsilon\to 1$. For conventional heterodyne detection in the limit $\eta\epsilon\to 1$, (5.17) implies that $\langle\boldsymbol{\alpha}|\rho|\boldsymbol{\alpha}\rangle/\pi^N$ is the classical probability density for the observation (5.18), i.e., conventional heterodyne detection is characterized by the POM $|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|$ when $\eta\epsilon\to 1$.

Two final notes about TCS-heterodyne detection are in order. In practice it may be impossible to build an image-oscillator beamsplitter which has *exactly* the unity intensity transmission coefficient for signal-band frequencies and *exactly* the unity intensity reflection coefficient for image-band frequencies. For a lossless beamsplitter with a signal-band intensity transmission coefficient ϵ_s and an image-band intensity reflection coefficient ϵ_i, we get (via

(5.16) and [15, sec. 3]) the characteristic functional

$$\phi_{dN''}(jv) = \exp\Big[-(2-\eta\epsilon-\epsilon_s) \cdot \int_{A_L} d\bar{x} \int_{\mathcal{T}_L} dt\, v^T(\bar{x},t)v(\bar{x},t)/4\eta\epsilon\epsilon_s\Big] \cdot \chi_A^{\rho_s}(\zeta^*,\zeta)\exp - \{\epsilon_i[|\nu|^2|\zeta|^2 + \mathrm{Re}(\zeta^\dagger\zeta^*\mu^*\nu)]/\epsilon_s\} \tag{5.19}$$

for TCS-heterodyne detection where $dN'' \equiv \epsilon_s^{-1/2} dN'$. When ϵ_s, ϵ_i, and $\eta\epsilon$ are all approximately equal to one, (5.19) collapses to $\chi_T^{\rho_s}(\zeta^*,\zeta)$, as expected. Should it prove unduly difficult to achieve $\epsilon_s \simeq 1$ and $\epsilon_i \simeq 1$, however, the POM $|\mu\nu\alpha\rangle\langle\mu\nu\alpha|$ can be realized by means of TCS-homodyne detection. In this configuration a 50/50 beamsplitter is used to mix a signal field of nominal frequency ω_0 with a TCS field of nominal frequency ω_0 and state $|\mu^*\nu^*\mathbf{0}\rangle$ as shown in Fig. 4. Two homodyne-detection processors are then used to simultaneously extract the cosine and sine quadrature information. That TCS-homodyne detection in the limit $1-\eta\epsilon\rightarrow 0$ measures the POM $|\mu\nu\alpha\rangle\langle\mu\nu\alpha|$ is easily established via Theorem 2; we omit the proof.

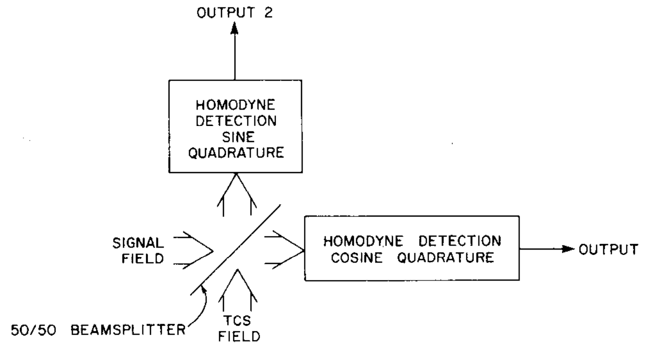

Fig. 4. Configuration for TCS-homodyne detection.

VI. Discussion

Before examining the utility of our results and speculating on their generalization, it is worthwhile to elaborate briefly on what we have accomplished so far. We have investigated the class of quantum measurements realizable using photoemissive detectors. Fundamentally, a quantum measurement is characterized by a probability operator measure (POM). Suppose there is an operator M whose eigenstates from the resolution of the identity associated with a particular POM and whose eigenvalues can be uniquely related to the observation values for this POM. We can then say that measurement of the operator M is the quantum measurement characterized by the POM under consideration. Additional descriptions of a quantum measurement result if the operator that is measured (in the sense just described) has a simple physical interpretation or if the measurement may be realized by a particular physical apparatus. It turns out that all of the measurements we have examined are amenable to all four of the proceeding descriptions: POM, operator measured, physical interpretation of the operator measured, and physical apparatus that realizes the measurement. These four approaches to characterizing our work are summarized in Table I for multimode fields, where $\eta\epsilon\rightarrow 1$, $\epsilon_s\rightarrow 1$, $\epsilon_i\rightarrow 1$, and $\phi(\bar{x},t) = -\omega_0 t$ are assumed throughout. In addition to their direct value as measurement realizations, the results summarized in Table I have significant applications of physical and communication interest, as indicated by the following examples.

Example 1—Multimode Field Measurements [26]

Photon detection is widely used to probe the quantum properties of a radiation field, for example, via photon correlation experiments. Such procedures, however, respond to many field modes at once. On the other hand, by means of post-detection filtering both homodyne and heterodyne detection can be made to respond to a single mode of the signal field (cf. (4.25)–(4.27)). Thus, that we now have the quantum statistics of these measurements means that we have techniques for probing the quantum characteristics of a multimode field one mode at a time.

Example 2—Analog Communication via Linear Modulation (cf. [16])

Suppose that a continuous complex-valued random variable $\gamma = \gamma_1 + j\gamma_2$ with given probability density function $p(\gamma^*, \gamma)$ is to be communicated by placing the nonvacuum mode of a single-mode radiation field in state ρ_γ. We shall assume that linear modulation is employed with an average energy constraint, i.e.,

$$\mathrm{tr}(\rho_\gamma a) = K\gamma \tag{6.1}$$

for K a positive constant and

$$\int p(\gamma^*,\gamma)\mathrm{tr}(\rho_\gamma a^\dagger a)\, d^2\gamma \leq N_s, \tag{6.2}$$

where a is the photon operator for the nonvacuum mode. The receiver will be assumed to use TCS-heterodyne detection, and its performance will be measured by a cost function

$$\bar{C} = \int d^2\gamma \int d^2\alpha\, p(\gamma^*,\gamma)\langle\mu\nu\alpha|\rho_\gamma|\mu\nu\alpha\rangle \cdot [(1-\zeta)(\alpha_1^2 - K^2\gamma_1^2) + \zeta(\alpha_2^2 - K^2\gamma_2^2)]/\pi K^2 E(|\gamma|^2) \tag{6.3}$$

where $0<\zeta<1$ is a weighting factor,[10] which is to be minimized by choice of μ, ν, K, and ρ_γ.

[10] For example, when $\zeta = E(\gamma_1^2)/E(|\gamma|^2)$ we find that $\bar{C}/(\zeta(1-\zeta)) = \sum_{i=1}^2 \{E[\mathrm{Var}(\alpha_i|\gamma)]/K^2E(\gamma_i^2)\}$, which is the sum of the reciprocal signal-to-noise ratios on the two quadratures. Minimizing $\bar{C}$ in this case is a reasonable performance objective in that it tends to equalize the mean-square estimation errors on the two quadratures at a minimum level.

TABLE I
SUMMARY OF QUANTUM MEASUREMENT RESULTS

Probability operator measure	$\lvert \underset{\sim}{n}\rangle\langle\underset{\sim}{n}\rvert$, where $\lvert\underset{\sim}{n}\rangle \equiv \bigotimes_m \lvert n_m\rangle$, $\lvert n_m\rangle$= number state	$\lvert\underset{\sim}{\alpha}_1\rangle_{11}\langle\underset{\sim}{\alpha}_1\rvert$, where $\lvert\underset{\sim}{\alpha}_1\rangle_1 \equiv \bigotimes_m \lvert\alpha_{1m}\rangle_1$, $\lvert\alpha_1\rangle_1$ = eigenstate of $a + a^\dagger$	$\lvert\underset{\sim}{\alpha}\rangle\langle\underset{\sim}{\alpha}\rvert$ $\lvert\underset{\sim}{\alpha}\rangle \equiv \bigotimes_m \lvert\alpha_m\rangle$ $\lvert\alpha\rangle$ = coherent state	$\lvert\mu\nu\underset{\sim}{\alpha}\rangle\langle\mu\nu\underset{\sim}{\alpha}\rvert$, where $\lvert\mu\nu\underset{\sim}{\alpha}\rangle \equiv \bigotimes_m \lvert\mu\nu\alpha_m\rangle$ $\lvert\mu\nu\alpha\rangle$ = TCS
Operator measured	$E^\dagger(\bar{x}, t)E(\bar{x}, t)$	$2^{-1}[E(\bar{x}, t)\, e^{j\omega_0 t} + E^\dagger(\bar{x}, t)\, e^{-j\omega_0 t}]$	$E(\bar{x}, t)\, e^{j(\omega_0+\omega_{IF})t}$	$\tilde{E}(\bar{x}, t)\, e^{j(\omega_0+\omega_{IF})t}$
Physical interpretation	photon flux density	single-quadrature field measurement	two-quadrature field measurement	TCS (canonical) measurement
Physical configuration	direct detection	homodyne detection	heterodyne detection	TCS-heterodyne detection

Note: In homodyne detection the signal field $E(x,t)$ is assumed to have nominal frequency ω_0, whereas in heterodyne and TCS-heterodyne detection it is assumed to have nominal frequency $\omega_0+\omega_{IF}$. In TCS-heterodyne detection with

$$E(\bar{x},t)=\sum_n a_n\xi_n(\bar{x},t)e^{-j\omega_{IF}t},$$

we define

$$\tilde{E}(\bar{x},t)=\sum_n (\mu a_n+\nu a_n^\dagger)\xi_n(\bar{x},t)e^{-j\omega_{IF}t}.$$

It is easily shown that

$$\bar{C}=\{(1-\zeta)[|\mu-\nu|^2+\mathrm{tr}[\bar{\rho}(\Delta a_1)^2]] + \zeta[|\mu+\nu|^2+\mathrm{tr}[\bar{\rho}(\Delta a_2)^2]]\} + 4K^2E(|\gamma|^2), \tag{6.4}$$

where $\bar{\rho}\equiv\int d^2\gamma p(\gamma^*,\gamma)\rho_\gamma$, which over all μ and ν has the minimum value

$$\bar{C}=\{(1-\zeta)\mathrm{tr}[\bar{\rho}(\Delta a_1)^2]+\zeta\mathrm{tr}[\bar{\rho}(\Delta a_2)^2] + 2[\zeta(1-\zeta)]^{1/2}\}+4K^2E(|\gamma|^2) \tag{6.5}$$

when μ and ν are real and satisfy (2.9) and

$$(1-\zeta)(\mu-\nu)^2=\zeta(\mu+\nu)^2. \tag{6.6}$$

For the sake of brevity we shall not optimize (6.5) over K and ρ_γ, although this can be accomplished without great difficulty. Instead, we shall assume that the ν value determined from (6.6) is less than $N_s^{1/2}$ and use the TCS transmitter $|\mu\nu(K\gamma)\rangle\langle\mu\nu(K\gamma)|$ with $K=[(N_s-|\nu|^2)/E(|\gamma|^2)]^{1/2}$. This system gives

$$\bar{C}=[\zeta(1-\zeta)]^{1/2}/K^2E(|\gamma|^2)\simeq[\zeta(1-\zeta)]^{1/2}/N_s, \tag{6.7}$$

when $N_s\gg|\nu|^2$. When $\zeta=\frac{1}{2}$, (2.9) and (6.6) yield $\mu=1$, $\nu=0$ so that conventional heterodyne detection is optimum and (6.7) yields the coherent state transmitter performance

$$\bar{C}=(2N_s)^{-1}. \tag{6.8}$$

When $\zeta\neq\frac{1}{2}$, a system that uses conventional heterodyne reception and a coherent state transmitter still has cost (6.8). Thus, in highly skewed situations wherein ζ or $(1-\zeta)$ is approximately zero, the TCS system greatly outperforms the coherent-state system.

Finally, let us comment on possible extension of our measurement realization theory. It should be clear that because Table I does not include the receivers of Kennedy [9] and Dolinar [10], [11] we have yet to exhaust the class of quantum measurements that one can make with available devices. Moreover, our theory, which has been limited to single-photon absorption devices, should be extended to incorporate existing multiphoton absorption devices [29] and their related homodyne, heterodyne [30], and TCS-heterodyne configurations.

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Comparison of Two Packet-Retransmission Techniques

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***Abstract*—Two methods of dealing with the overflow problem in packet switches are compared: switch-retransmission and host-retransmission. The comparison is based on approximate analytic models of tandem queueing networks and is verified by simulation.**

I. Introduction

A packet-switched communications network consists of packet switches interconnected by transmission links. Besides providing the switching function, packet switches connect subscribers (host computers) to the network. Subscriber messages are broken into packets, typically up to 255 characters long, and transmitted packet by packet between switches until the destination subscriber is reached. Of course, each packet has to carry information about its destination in the header. Buffering techniques are employed to hold packets which may have to queue in a switch until forwarded to the next switch or the destination subscriber.

Packet-switched networks have usually been modelled as networks of queues, assuming infinite waiting room and consequently no traffic loss. On the other hand, real packet switches have finite waiting room, typically sufficient for about 20 packets. Thus buffer overflow is possible, particularly under heavy traffic. Our objective is to investigate strategies for dealing with packets rejected because of overflow.

The most common technique, here called *switch-retransmission*, is used in the ARPA network, in which, if a packet cannot be accepted by a switch, it is retransmitted from a backup copy held in the preceding switch [6]. Another technique which we call *host-retransmission* is used in the Cyclades network [7]. Here the network drops a packet which arrives at a full switch to be resent later by the source subscriber. It has often been claimed that the switch-retransmission method is superior. Our analysis will show that this is true, but that the difference is slight under certain conditions. In these cases, other considerations will be required to determine the best choice.

We will consider a single source destination path with no interfering traffic which can be modelled as a tandem network of queues. In Section II our models are formulated; they are analyzed in Sections III and IV, with numerical examples given in Section V.

Manuscript received October 2, 1977; revised April 27, 1979.
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II. Tandem Networks with Retransmission

We consider two models, switch-retransmission and host-retransmission, as shown in Figs. 1 and 2. For both models we make the following assumptions.

- External arrivals occur only at the initial station 0 which represents the host computer; they constitute a Poisson process of rate λ.
- The packet length is an exponential random variable. The transmission or retransmission time of a packet is proportional to its length; hence the service time for a link n is an exponential random variable with mean $1/\mu_n$. For the purpose of analysis, Kleinrock's independence assumption is in effect [4]; a new packet length is chosen from an exponential distribution in each station. Moreover, we assume that at the same station each retransmission (if there are any) operates on an independently generated realization of the packet length.
- While station 0 has infinite waiting room, each remaining station $n>0$ can queue only up to M_n packets.
- A packet which arrives at a full station is refused. The probability of packet refusal at station $n>0$ is denoted by p_n. The destination subscriber never refuses a packet; we define $p_{N+1}=0$.

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