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Generalized multistability and chaos in quantum optics

By F. T. Arecchi

Department of Physics, University of Florence, Florence, Italy, and
Istituto Nazionale di Ottica, Largo Enrico Fermi 6, 50125 *Arcetri-Firenze, Italy*

Three experimental situations for CO_2 lasers (a laser with modulated losses, a ring laser with competition between forward and backward waves, and a laser with injected signal) are analysed as examples of the onset of chaos in systems with a homogeneous gain line and with a particular timescale imposed by the values of the relaxation constants.

We stress the coexistence of several basins of attraction (generalized multistability) and their coupling by external noise. This coupling induces a low-frequency branch in the power spectrum. Comparison is made between the spectra of noise-induced jumps over independent attractors and the spectrum of deterministic diffusion within subregions of the same attractor. At the borderline between the two classes of phenomena a scaling law holds, relating the control parameter and the external noise in their effect on the mean escape time from a given stability region.

This is a review of a research line at I.N.O. dealing with the role of noise in multistable dynamical systems. The main point is as follows. For some ranges of the control parameters, there can be the simultaneous coexistence of more than one attractor. The basins of attraction are disjoint; hence, provided that the attractors are structurally stable, there is no connection between them. The addition of noise acts as an added dimension to the phase space, which allows a bridging of the attractors. This bridging is evidenced by the appearance of low-frequency tails in the power spectrum, which resemble the so-called $1/f$ noise. Experimental evidence of this effect was given in (*a*) electronic nonlinear devices (Arecchi & Lisi 1982; Arecchi & Califano 1984) such as a Duffing oscillator driven by external modulation (figure 1); (*b*) a CO_2 laser with modulated losses (Arecchi *et al.* 1982) (figure 2); (*c*) a CO_2 laser in a ring configuration implying a competition between the two counter-propagating fields via scattering through the nonlinear inversion grating (Tredicce *et al.* 1984 and this symposium); (*d*) a CO_2 laser with an injected signal at a frequency different from the free-running laser frequency (Arecchi *et al.* 1984*c*). The theory of noise-induced jumps among many attractors was given at two levels: (*e*) with reference to a one-dimensional cubic iteration map, which is the first generalization of the logistic map allowing for more than one attractor (Arecchi *et al.* 1984*a*); (*f*) by a numerical study of the parameter space of a Duffing equation (Arecchi *et al.* 1984*b*) with particular emphasis on the transition region where the régime of two separate attractors undergoes crisis (Grebogi *et al.* 1982). This case is particularly relevant in that it allows comparison with a completely different phenomenon, i.e. the deterministic diffusion within subregions of the same attractor (Geisel & Nierwetberg 1982; Grossmann & Fujisaka 1982) (see figures 3 and 4).

Around the crisis (transition from noise-induced jumps to deterministic diffusion) a universal scaling law relates the control parameter and the external noise in their effect on the mean

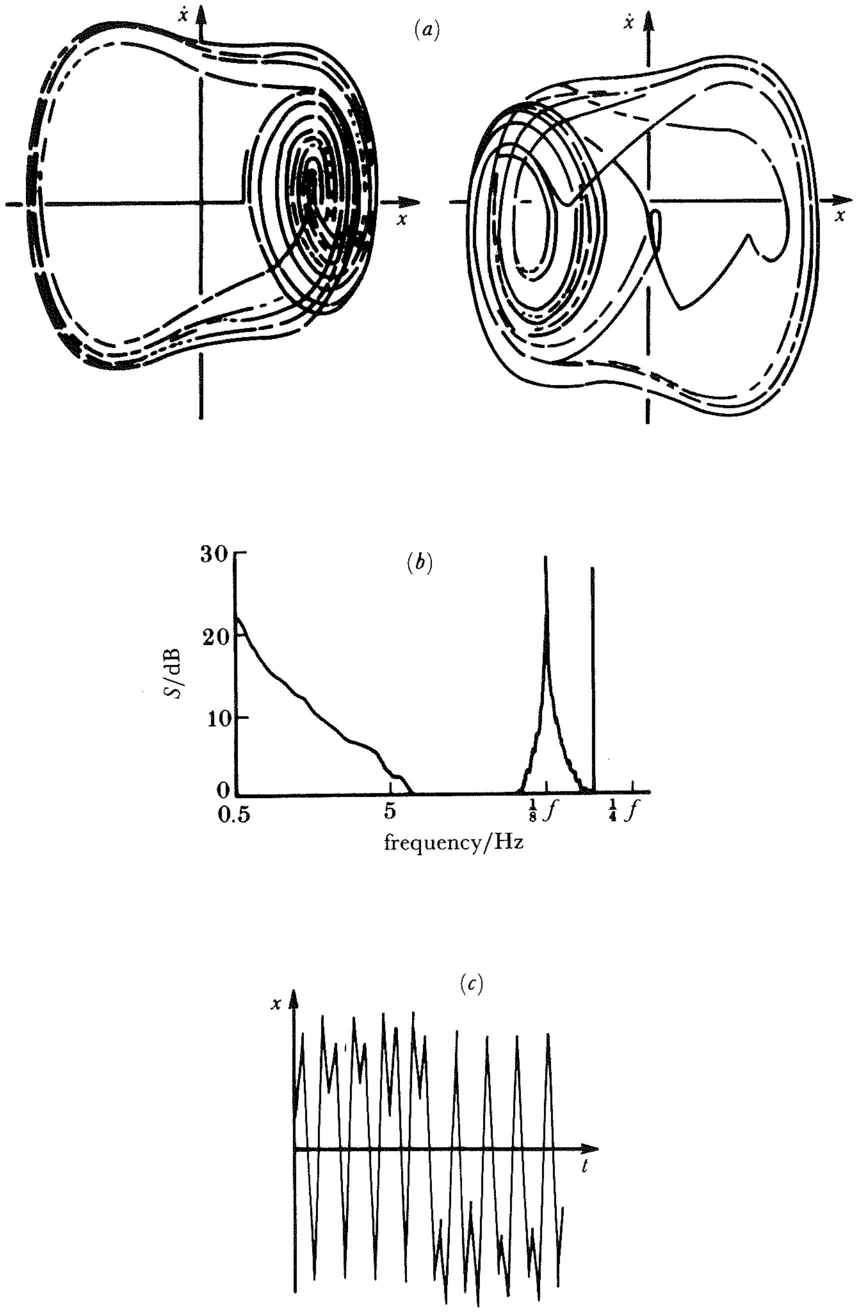

FIGURE 1. Experimental generalized multistability in an analogue system with an electronic nonlinearity studied by Arecchi & Lisi (1982): (a) phase plane (x, $\dot{x}$); (b) power spectrum; (c) signal x against time.

escape time from a given stability region. The main experimental outcome of these phenomena is the appearance of low-frequency tails in the power spectra, which display a $f^{-\alpha}$ behaviour. The slope α in a double-logarithmic plot has been measured (Arecchi & Lisi 1982; Arecchi & Califano 1984; Arecchi *et al.* 1982; Tredicce *et al.* 1984 and this symposium) to vary between 0.6 and 1.7 and has been proved theoretically (Arecchi *et al.* 1984*a*) to depend on the number of attractors among which jumps occur and on the Lyapunov exponents of the attractors.

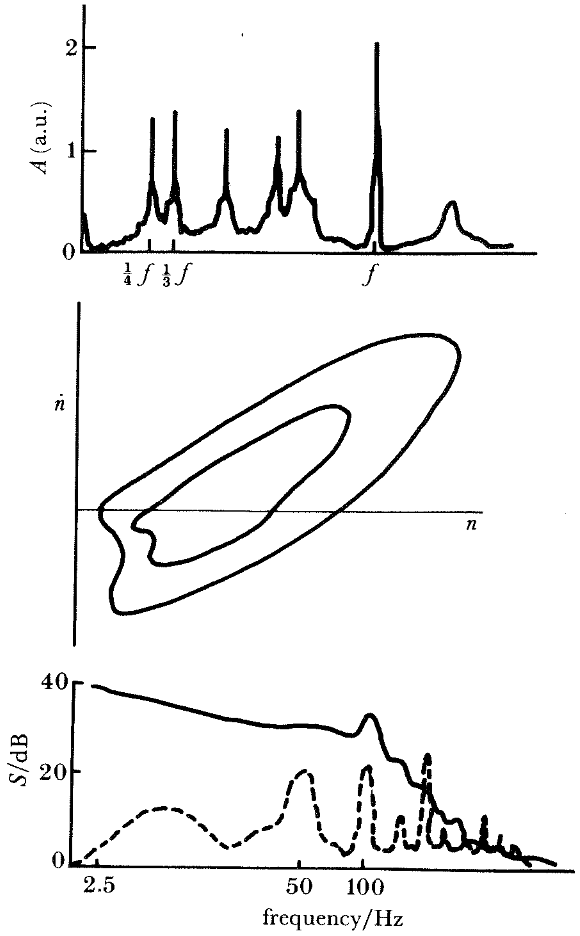

FIGURE 2. Experimental generalized multistability in a CO_2 laser with modulated losses studied by Arecchi *et al.* (1982): (*a*) two superposed ($\frac{1}{3}f$ and $\frac{1}{4}f$) power spectra; (*b*) two coexisting attractors; (*c*) low-frequency part of the spectrum with noise (solid line) and without noise (broken line).

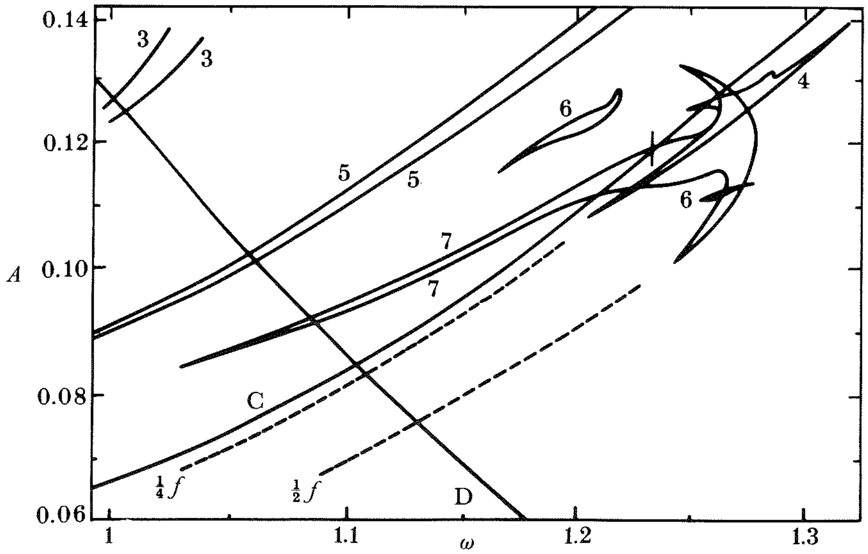

FIGURE 3. Coexistence of many basins of attraction from numerical studies of the Duffing oscillator equations (Arecchi *et al.* 1984*b*) in the phase space of the amplitude (A) and the frequency (ω) of the modulation. Numbers denote the periodicity of the attractors. The vertical line in the parameter space indicates the range over which A is changed in figure 4.

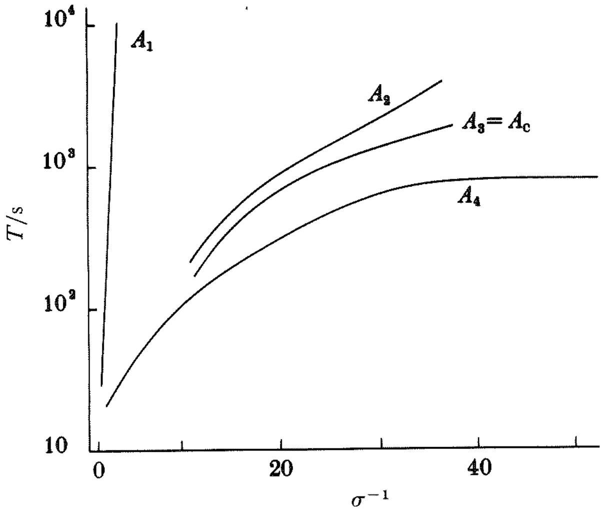

FIGURE 4. The mean escape time from the period-7 attractor plotted against the amplitude of the applied noise (Arecchi *et al.* 1984*b*) A_1 and A_2 correspond to conditions in which the attractor is stable against infinitesimal perturbations. $A_3 \equiv A_c$ corresponds to the frontier of the deterministic destabilization; A_4 is beyond the crisis.

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