

### Coherent States for $r$ -Level Atoms (\*).

F. T. ARECCHI

*Istituto di Fisica dell'Università - Pavia  
Laboratori C.I.S.E. - Milano*

R. GILMORE

*Physics Department, University of South Florida - Tampa, Fla.*

D. M. KIM

*Department of Electrical Engineering, Rice University - Houston, Tex.*

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It is well known that a 2-level system can be described by the  $su_2$  angular-momentum algebra (1). The Hamiltonian for a system with two internal degree of freedom is

$$(1) \quad \begin{cases} H = H_0 + H_p, \\ H_0 = \Delta E S_z, \\ H_p = \lambda(t) S_+ + \lambda^*(t) S_- . \end{cases}$$

Here  $\Delta E$  is the separation between the ground and excited states,  $\lambda(t)$  describes the coupling of the 2-level system to an external field, and  $S_{\pm}$ ,  $S_z$  obey the  $su_2$  commutation relations

$$(2) \quad [S_z, S_{\pm}] = \pm S_{\pm}, \quad [S_+, S_-] = 2S_z .$$

When the external driving field is classical,  $\lambda(t)$  is a complex  $c$ -number function of time. If each of  $N$  identical 2-level atoms interacts with the same classical external driving field, the hamiltonian describing the ensemble is obtained by replacing  $S$  by  $\sum_{t=1}^N (S)_t = J$  in (1), where  $(S)_t$  describes the dynamics of atom  $t$ . The  $J$  also obey the

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(1) R. H. DICKE and J. P. WITTKER: *Introduction to Quantum Mechanics* (Reading, Mass., 1960).

$su_2$  commutation relations. Transitions between eigenstates  $|jm\rangle$  of  $H_0$  can exhibit super-radiance<sup>(2)</sup>. That is, in certain states the transition intensity is proportional to the square of the number  $N$  of atoms:

$$I \simeq (N/2)^2 I_0, \quad j \simeq N/2, \quad m \simeq 0,$$

where  $I_0$  is the corresponding single-atom transition intensity. The eigenstates  $|jm\rangle$  are often called « Dicke states », since superradiance was first discussed by DICKE<sup>(2)</sup>. Under the perturbation  $H_p$  the ensemble will evolve from the ground state  $|j, -j\rangle$  into an atomic coherent state. In terms of a Dicke basis, the atomic coherent states are given by a binomial expansion

$$(3) \quad \left| \begin{matrix} j \\ x_1, x_2 \end{matrix} \right\rangle = U(x) \left| \begin{matrix} j \\ -j \end{matrix} \right\rangle = \sum_{m=-j}^{+j} \left| \begin{matrix} j \\ m \end{matrix} \right\rangle \left\{ \frac{N!}{(j+m)!(j-m)!} \right\}^{\frac{1}{2}} (x_1)^{j-m} (x_2)^{j+m}.$$

For  $j = \frac{1}{2}$  these states are identical to the NMR electron states, first considered by BLOCH<sup>(3)</sup> (set  $x_1 = \cos \frac{1}{2}\theta$ ,  $x_2 = \exp[-i\varphi] \sin \frac{1}{2}\theta$ ). As a result, the states (3) are often called « Bloch states ». The Dicke and Bloch states for 2-level systems are entirely analogous to the Fock<sup>(4)</sup> (diagonal) and Glauber<sup>(5)</sup> (coherent) states for a single mode of the electromagnetic field. This analogy has been made manifest by a group contraction process<sup>(6)</sup>.

Dicke and Bloch states for 2-level atoms have been useful in the description of physical problems. However, it is valid to describe a physical system as a 2-level system only when a resonance approximation can be made. Off resonance, additional internal degrees of freedom must be taken into account. As a result, it is useful to extend the  $SU_2$  formalism for 2-level atoms to an  $SU_r$  formalism for  $r$ -level atoms. We will construct the analogs for  $r$ -level atoms of the Dicke and Bloch states for 2-level atoms.

The Hamiltonian describing a single system with  $r$  internal states of energies  $E_1 < E_2 < \dots < E_r$  is

$$(4) \quad \left\{ \begin{array}{l} H = H_0 + H_p, \\ H_0 = \mathbf{v} \cdot \mathbf{H}, \quad H_i = \mathbf{u}_i^i, \\ H_p = \sum_{i=1}^r \sum_{j>i} \lambda_i^j(t) \mathbf{u}_i^j + \text{Adjoint.} \end{array} \right.$$

$\mathbf{v}$  is an  $r$ -dimensional vector with components  $E_1, E_2, \dots, E_r$ . The operators  $\mathbf{u}_i^j$ , describing transformations from the  $i$ -th state to the  $j$ -th state, obey the  $u(r)$  commutation relations<sup>(7)</sup>

$$(5) \quad [\mathbf{u}_i^j, \mathbf{u}_s^t] = \mathbf{u}_s^j \delta_i^t - \mathbf{u}_i^t \delta_s^j.$$

$\lambda_i^j(t)$  describes the coupling of the  $j$  and  $i$  states to an external driving field;  $\lambda_i^j(t)$  is a complex  $c$ -number function of time when the external field is assumed classical. The Hamiltonian describing  $N$  indistinguishable  $r$ -level atoms interacting only with the

(2) R. H. DICKE: *Phys. Rev.*, **93**, 99 (1954).

(3) F. BLOCH: *Phys. Rev.*, **70**, 460 (1946).

(4) V. FOCK: *Zeit. für Phys.*, **75**, 622 (1932).

(5) R. J. GLAUBER: *Phys. Rev.*, **130**, 2529 (1963); **131**, 2766 (1963).

(6) F. T. ARECCHI, E. COURTENS, R. GILMORE and H. THOMAS: *Phys. Rev. A*, **6**, 2211 (1972).

(7) When  $H$  is traceless, the  $U_i^j$  obey  $SU_r$  commutation relations.

same external driving field is obtained by replacing  $\mathbf{u}_i^j \rightarrow \sum_{t=1}^N (\mathbf{u}_i^j)_t = U_i^j$  in (4). The  $U_i^j$  obey the commutation relations given in (5).

The eigenstates of  $H_0$  are the Gel'fand-Tsetlein states (<sup>8-10</sup>). These belong to a  $U(r)$ -invariant subspace described by a Young partition  $\bar{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_r \geq 0)$ . Matrix elements for the transition operators  $U_i^j$  between « Dicke » states can be written down explicitly (<sup>8-10</sup>). Superradiance (<sup>11</sup>) is possible, but it is confined primarily to the fully symmetric subspace  $\{N, \hat{0}\}$ . As in the 2-level case, the intensity of a particular transition can never exceed  $(N/2)^2$  times the corresponding single-atom transition intensity.

Under the influence of  $H_p$ , the ensemble will evolve from the ground state to a linear superposition of states belonging to the space  $\{N, \hat{0}\}$ , since  $\bar{\lambda}$  is a constant of the motion and the total-system ground state lies in the fully symmetric subspace  $\{N, \hat{0}\}$ . The eigenstates of  $H_0$  within  $\{N, \hat{0}\}$  are described by a sequence of integers  $N = n_r \geq n_{r-1} \geq \dots \geq n_2 \geq n_1 \geq 0$ , where  $n_i$  is the length of the single-row Young partition at level (<sup>10</sup>)  $i$  in the Gel'fand-Tsetlein basis vector. Under  $H_p$ , the ground state  $|\{N, \hat{0}\}, N, N, \dots, N\rangle$  evolves into a coherent atomic state (« Bloch » state) given explicitly by an  $r$ -nomial expansion:

$$(6) \quad |\text{gnd}\rangle = \left| \begin{array}{c} \{N, \hat{0}\} \\ N, N, \dots, N \end{array} \right\rangle \xrightarrow{H_p} U(x) \left| \begin{array}{c} \{N, \hat{0}\} \\ N, N, \dots, N \end{array} \right\rangle = \\ = \left| \begin{array}{c} \{N, \hat{0}\} \\ x_1, x_2, \dots, x_r \end{array} \right\rangle \sum \left| \begin{array}{c} \{N, \hat{0}\} \\ n_r, \dots, n_2, n_1 \end{array} \right\rangle \left\{ \frac{N!}{(N - n_{r-1})! \dots (n_2 - n_1)! n_1!} \right\}^{\frac{1}{2}} (x_1)^{n_1} (x_2)^{n_2 - n_1} \dots (x_r)^{N - n_{r-1}}.$$

The time-dependent coefficients  $x_i(t)$  obey

$$x_1^2 + |x_2|^2 + \dots + |x_r|^2 = 1.$$

The  $x_i$  are complex, except for  $x_1$ , which can always be chosen real. The  $r$ -level atomic coherent states therefore exist in 1-1 correspondence with the points on the surface of the unit sphere  $S^{2(r-1)}$  of dimensionality  $2(r-1)$ .

These  $r$ -level atomic coherent states possess all the properties which the more familiar 2-level atomic coherent states possess (<sup>6</sup>). They are nonorthogonal and overcomplete:

$$(7) \quad \left\langle \begin{array}{c} \{N, \hat{0}\} \\ y \end{array} \middle| \begin{array}{c} \{N, \hat{0}\} \\ x \end{array} \right\rangle = \left\{ \sum_{i=1}^r y_i^* x_i \right\}^N,$$

$$(8) \quad \int \left| \begin{array}{c} \{N, \hat{0}\} \\ x \end{array} \right\rangle d\mu(x) \left\langle \begin{array}{c} \{N, \hat{0}\} \\ x \end{array} \right| = \frac{\text{Vol}[S^{2(r-1)}]}{\dim\{N, \hat{0}\}} I_{\{N, \hat{0}\}}.$$

Here  $d\mu(x)$  is the measure on  $S^{2(r-1)}$ :

$$(9) \quad \begin{cases} d\mu(x) &= \delta([x_1^2 + |x_2|^2 + \dots + |x_r|^2]^{\frac{1}{2}} - 1) dx_1 d^2x_2 \dots d^2x_r, \\ \text{Vol}[S^{2(r-1)}] &= 2\pi^{r-\frac{1}{2}}/\Gamma(r-\frac{1}{2}), \\ \dim\{N, \hat{0}\} &= (r+N-1)!/N!(r-1)!. \end{cases}$$

(<sup>8</sup>) I. M. GEL'FAND and M. L. TSETLEIN: *Doklady Akad. Nauk SSSR*, **71**, 825 (1950).

(<sup>9</sup>) G. E. BAIRD and L. C. BIEDENHARN: *Journ. Math. Phys.*, **4**, 1449 (1963).

(<sup>10</sup>) R. GILMORE: *Journ. Math. Phys.*, **11**, 3420 (1970).

(<sup>11</sup>) L. A. SHELEPIN: *Sov. Phys. JETP*, **27**, 784 (1968).

Coherent states obey three kinds of « eigenvalue » equations. These can be determined by applying a similarity transformation to the eigenvalue equations which define the ground state, itself a coherent state with  $x_1 = 1$ ,  $x_i = 0$ ,  $i = 2, 3, \dots, r$ . The eigenvalue equations which determine the ground state are

$$(10) \quad \begin{cases} (\text{Cas. Op.})_k |\text{gnd}\rangle = \widehat{\text{Cas.}} |\text{gnd}\rangle, \\ U_i^i |\text{gnd}\rangle = (n_i - n_{i-1}) |\text{gnd}\rangle, \\ U_{i+1}^i |\text{gnd}\rangle = 0 |\text{gnd}\rangle. \end{cases}$$

Here  $(\text{Cas. Op.})_k$  is the  $k$ -th Casimir invariant for  $U(r)$ , and  $\widehat{\text{Cas.}}$  is its eigenvalue <sup>(12)</sup>. The similarity transform leaves  $(\text{Cas. Op.})_k$  unaffected, as well as the eigenvalues on the right-hand side. On the left we have

$$U(x) U_j^i U^{-1}(x) = M_i^i(x) U_i^i M_j^s(x)^*,$$

with

$$M_1^1(x) = x_1, \quad M_i^i(x) = x_i = -M_1^i(x)^*, \quad M_i^r(x) = [I_{r-1} - x_i x_j^*]_{\text{rt.}}^{\frac{1}{2}}.$$

Next, we construct a generating function <sup>(6)</sup> for computing matrix elements of the operators  $U_j^i$  between two coherent states. This can be done most easily by using the appropriate  $U(r)$  BCH relation <sup>(6)</sup>. The result is (under assumption  $e_1^1 = f_1^1 = 0$ )

$$(11) \quad \left\langle \begin{matrix} \{N, \hat{0}\} \\ y \end{matrix} \middle| \exp[e_1^1 U_1^1] \exp[f_1^k U_1^k] \middle| \begin{matrix} \{N, \hat{0}\} \\ x \end{matrix} \right\rangle = \{y_1 x_1 + (y_1 e_k^1 + y_k^*) (f_1^k x_1 + x_k)\}^N.$$

This entire discussion can be carried out for coherent states within other than the fully symmetric representation. This must be done when the system evolves from a metastable ground state. A semi-classical description has already been given for 2-level atoms <sup>(13)</sup>. In the state space with  $\bar{\lambda} = \{\lambda_1, \lambda_2, \dots, \lambda_p \neq 0, \lambda_p = \dots = 0\}$ , the ground state is the Gelfand-Tsetlin state of highest weight, and the coherent state is

$$(12) \quad \left| \begin{matrix} \bar{\lambda} \\ X \end{matrix} \right\rangle = \exp[A_i^\alpha U_i^\alpha + A_\beta^j U_\beta^j] \left| \begin{matrix} \bar{\lambda} \\ \bar{M}^h \end{matrix} \right\rangle$$

with  $A_i^\alpha = -A_\alpha^{i*}$ ,  $1 \leq i, j, k, \dots \leq p$ ,  $p+1 \leq \alpha, \beta, \gamma, \dots \leq p+q = r$ , and  $X_i^\alpha = A_j^\alpha \cdot (\sin \sqrt{A^+ A} / \sqrt{A^+ A})_i^j$ . The results (6)-(11) hold for the coherent states (12), provided the obvious modifications are made. For example, the generating functions for matrix elements of the operators  $U$  are

$$\begin{aligned} \left\langle \begin{matrix} \bar{\lambda} \\ Y \end{matrix} \middle| \exp[E_\beta^j U_\beta^j] \exp[F_i^\alpha U_i^\alpha] \middle| \begin{matrix} \bar{\lambda} \\ X \end{matrix} \right\rangle &= \prod_{k=1}^p \{M_k(E, F)\}^{\lambda_k - \lambda_{k+1}}, \\ \left\langle \begin{matrix} \bar{\lambda} \\ Y \end{matrix} \middle| \exp[S_\beta^\alpha U_\beta^\alpha] \exp[T_j^i U_j^i] \middle| \begin{matrix} \bar{\lambda} \\ X \end{matrix} \right\rangle &= \prod_{k=1}^p \{M_k(S, T)\}^{\lambda_k - \lambda_{k+1}}, \end{aligned}$$

<sup>(12)</sup> R. GILMORE: *Journ. Math. Phys.*, **11**, 1855 (1970); *Ann. of Phys.* (to appear).

<sup>(13)</sup> C. R. STROUD jr., J. H. EBERLY, W. L. LAMA and L. MANDEL: *Phys. Rev. A*, **5**, 1094 (1972).

where  $M_k(-, -)$  is the determinant of the diagonal minor of  $M(-, -)$  containing rows  $1, 2, \dots, k$  and columns  $1, 2, \dots, k$ , and the matrices  $M(-, -)$  are given by

$$\begin{aligned} M_j^i(E, F) &= Z_k^i(Y) Z_j^k(X) + \{-Y_\alpha^i + Z_k^i(Y) E_\alpha^k\} \{X_j^\alpha + F_i^\alpha Z_j^i(X)\}, \\ M_j^i(S, T) &= Z_k^i(Y) \{e^S\}_i^k Z_j^i(X) + Y_i^{\alpha*} \{e^S\}_{\alpha\beta} X_j^\beta, \\ Z(X) &= [I_p - X^\dagger X]^{\frac{1}{2}} = Z^\dagger(X). \end{aligned}$$

We have described the important mathematical properties of coherent states for ensembles of systems with  $r$ -internal degrees of freedom. The physical importance of these states is this: under a classical external driving field, and neglecting radiation reaction, spontaneous emission, and particle-particle interactions, a coherent state will evolve into a coherent state. This holds in particular if the system evolves from its ground state.