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Theory of Optical Maser Amplifiers

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***Abstract*—The interaction between an electromagnetic (em) pulse and a maser medium is described by a general set of five equations, under the assumption of a homogeneously broadened electric-dipole transition with two Bloch relaxation times T_2 and T_1, and of a linear broadband loss mechanism. When the equations are specialized at resonance, their solutions include the results of the previous treatments on the amplifier problem obtained under particular assumptions. The steady-state pulse (S. S. P.) introduced by Wittke and Warter for $T_2/T_1 = 0$ is here generalized for $T_2/T_1 \neq 0$ and it is shown to propagate at the same velocity of the light in the medium. In the case $T_2/T_1 = 0$ the steady state is described by exact analytical relations. For times short in comparison to the relaxation times, a solution is given which generalizes the usual interaction formula between an em field and a two-level system by introducing propagation effects. In the general case out of resonance, it is shown that an S. S. P. exists, and that its frequency coincides with the frequency of the atomic transition, independent of the frequency of the input field.**

I. Introduction

THIS PAPER gives a theoretical description of the interaction between a maser medium and an em pulse propagating in it.

A suitable model for this problem is an ensemble of identical two-level systems, which at some time starts interacting with a linearly polarized quasi-monochromatic wave. The transition is electric dipole with moment μ and is assumed homogeneously broadened. A certain degree of randomness must be introduced to take into account the incoherent pumping mechanism and the interactions with the surrounding media. Therefore the ensemble is described by a density matrix with two relaxation times, T_1 for the diagonal elements, and T_2 for the off-diagonal elements [1], analogous to the phenomenological relaxation times used by Bloch in nuclear magnetic resonance [2]. Assume the field to be intense enough to be described classically; it is understood as a quasi-monochromatic plane wave. The evolution of the active medium is accounted for by three coupled equations in the density matrix components. Furthermore, in the presence of a distributed linear loss mechanism of such a broadband that does not contribute dispersion effects, the field is described by two transport equations.

This basic set of five equations is first specialized at resonance, in order to make a comparison with some recent works dealing with the same subject [3]–[5]. In Frantz and Nodvik [3] and DuBois [4] a rate-equation approach was used; we will show how it gives a correct picture for observation times much longer than the coherence time T_2 of the induced dipoles. A density matrix treatment was made by Wittke and Warter [5] under the foregoing assumptions for the amplifying medium and the so-called S.S.P. was introduced as the asymptotic limit at which the em pulse no longer changes for an observer moving with the pulse. However, Wittke and Warter limit their treatment to resonance and use a finite difference field equation which allows for only numerical computations in the limiting case of $T_2/T_1 = 0$ (recovery time of inverted population much longer than the dipole coherence time).

In this paper, both rate equations and S.S.P. treatments appear as asymptotic evaluations over different regions of a general space-time domain. Furthermore, the S.S.P. is treated in the general case $T_2/T_1 \neq 0$, and its behavior has quite different results from those of Wittke and Warter [5], especially since this behavior propagates at the same

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velocity of the light in the medium. At resonance, a simple solution can be given for times short in comparison to relaxation times, and it generalizes the usual interaction formula between an em field and a two-level system by introducing propagation effects. The general asymptotic solution out of resonance shows that even in this case a S.S.P. exists, and its frequency coincides with the frequency of the atomic transition, independently of the frequency of the input field.

II. Interaction Equations

To further specify the general assumptions, we think of the amplifying medium as a system of identical atoms uniformly distributed along an x direction and weakly coupled among each other, so that mutual interaction can be accounted for by the relaxation time T_2, which is the reciprocal of the homogeneous linewidth. The electric field $E(x, t)$ starts at a time that we assume as origin $t = 0$ at the boundary $x = 0$ of the material, and propagates as a quasi-monochromatic linearly polarized plane wave for $x > 0$

$$E(x, t) = E_0(x, t) \cos [\omega t - kx + \varphi(x, t)], \tag{1}$$

where ω and $k = \omega/c$ are constants, $c = (\epsilon\mu_0)^{-1/2}$ being the light velocity in the host material which contains the active two-level systems. The amplitude E_0 and phase φ are slowly varying with respect to the optical frequency changes, that is,

$$\begin{aligned} &\left|\frac{\partial E_0}{\partial x}\right| \ll kE_0; \qquad \left|\frac{\partial E_0}{\partial t}\right| \ll \omega E_0, \\ &\left|\frac{\partial \varphi}{\partial x}\right| \ll k; \qquad \left|\frac{\partial \varphi}{\partial t}\right| \ll \omega. \end{aligned} \tag{2}$$

The evolution of the atomic ensemble under the influence of this field is given by (47) of Appendix I. In the presence of the field, the medium is characterized by a macroscopic volume polarization with "in-phase" and "in-quadrature" components with respect to $E(x, t)$

$$\begin{aligned} P(x, t) = {} & C(x, t) \cos [\omega t - kx + \varphi(x, t)] \\ & + S(x, t) \sin [\omega t - kx + \varphi(x, t)], \end{aligned} \tag{3}$$

and by a population difference D between upper and lower level. C, S, and D are connected to the atomic parameters through relations (48). The propagation of the field with induced polarization as a source term is given by (51) of Appendix II.

A more suitable presentation for this transport problem is obtained in a transformed frame characterized by

$$\tau = (t - x/c)/T_2$$

$$\eta = x/(cT_2).$$

Furthermore, let us normalize the induced polarization components and the inverted population to the value D_e of the population difference at equilibrium with the pump, and write the field in an adimensional form. This gives us a set of useful parameters

$$\begin{aligned} \Theta &= T_2 E_0 \mu/\hbar = T_2 \Lambda \\ \Delta &= D/D_e \\ \Phi &= S/(\mu D_e) \\ \Psi &= -C/(\mu D_e). \end{aligned} \tag{4}$$

The first three quantities have been denoted as the similar ones of Wittke and Warter [5] for sake of comparison. Furthermore, this treatment considers Ψ and φ which account for the dispersion properties of the medium. In this new notation, the whole set of interaction equations becomes

$$\begin{aligned} \frac{\partial \Phi}{\partial \tau} &= -\Phi + \Theta\Delta - \left(\delta_0 + \frac{\partial \varphi}{\partial \tau}\right)\Psi \\ \frac{\partial \Psi}{\partial \tau} &= -\Psi + \left(\delta_0 + \frac{\partial \varphi}{\partial \tau}\right)\Phi \\ \frac{\partial \Delta}{\partial \tau} &= -\Theta\Phi - \gamma(\Delta - 1) \\ \Theta \frac{\partial \varphi}{\partial \eta} &= G\Psi \\ \frac{\partial \Theta}{\partial \eta} &= G(\Phi - L\Theta), \end{aligned} \tag{5}$$

in which

$$\delta_0 = \omega - \omega_0$$

is the difference between the frequency ω of the field and the frequency ω_0 of the atomic transition,

$$\gamma = T_2/T_1 \tag{6}$$

is the ratio of the two relaxation times, and

$$G = \alpha_g c T_2/2 \tag{7}$$

is the field gain over length cT_2. Here we have denoted by

$$\alpha_g = -\frac{\omega\mu^2 N}{\epsilon c\hbar} D_e T_2 \tag{8}$$

the power-gain coefficient per unit length when the population is at equilibrium with the pump. The minus sign derives from the fact the D_e has been defined as the difference between lower and upper level populations, and is negative when the population is inverted. In the following, α_g may be calculated at ω_0 even when the field is at ω, because this slight difference is immaterial. Finally,

$$L = \frac{\sigma T_2}{2\epsilon G} \tag{9}$$

is the ratio between the losses and the gain G.

III. Discussion of the Amplifier Equations

Equations (5) give the behavior of the system in the region $\eta \geq 0$, $\tau \geq 0$ provided the following boundary conditions are satisfied

$$
\begin{aligned}
\Phi(\eta,\ \tau = 0) &= 0\\
\Psi(\eta,\ \tau = 0) &= 0\\
\Delta(\eta,\ \tau = 0) &= 1\\
\Theta(\eta = 0,\ \tau) &= \Theta_0\\
\varphi(\eta = 0,\ \tau) &= 0.
\end{aligned}
\tag{10}
$$

These conditions state that for $\tau = 0$ the active medium is in equilibrium with the pump and that the envelope of the applied field at the amplifier input is a step function of strength Θ_0. By a series expansion around the point $\eta = \tau = 0$ one can show that these conditions are sufficient to determine a unique solution. First we investigate the problem in the case $\delta_0 = \omega - \omega_0 = 0$, that is, incoming field at resonance with the atomic line. For $\delta_0 = 0$ one immediately verifies by direct substitution that a possible solution, and therefore the only solution consistent with the stated conditions is

$$\Psi = 0, \qquad \varphi = 0$$

over the whole $\eta - \tau$ domain.

Equations (5) reduce, therefore, to

$$
\begin{aligned}
\frac{\partial \Phi}{\partial \tau} &= -\Phi + \Theta\Delta\\
\frac{\partial \Delta}{\partial \tau} &= -\Phi\theta - \cancel{\Theta}\gamma[\Delta - 1]\\
\frac{\partial \Theta}{\partial \eta} &= G[\Phi - \ \Theta].
\end{aligned}
\tag{11}
$$

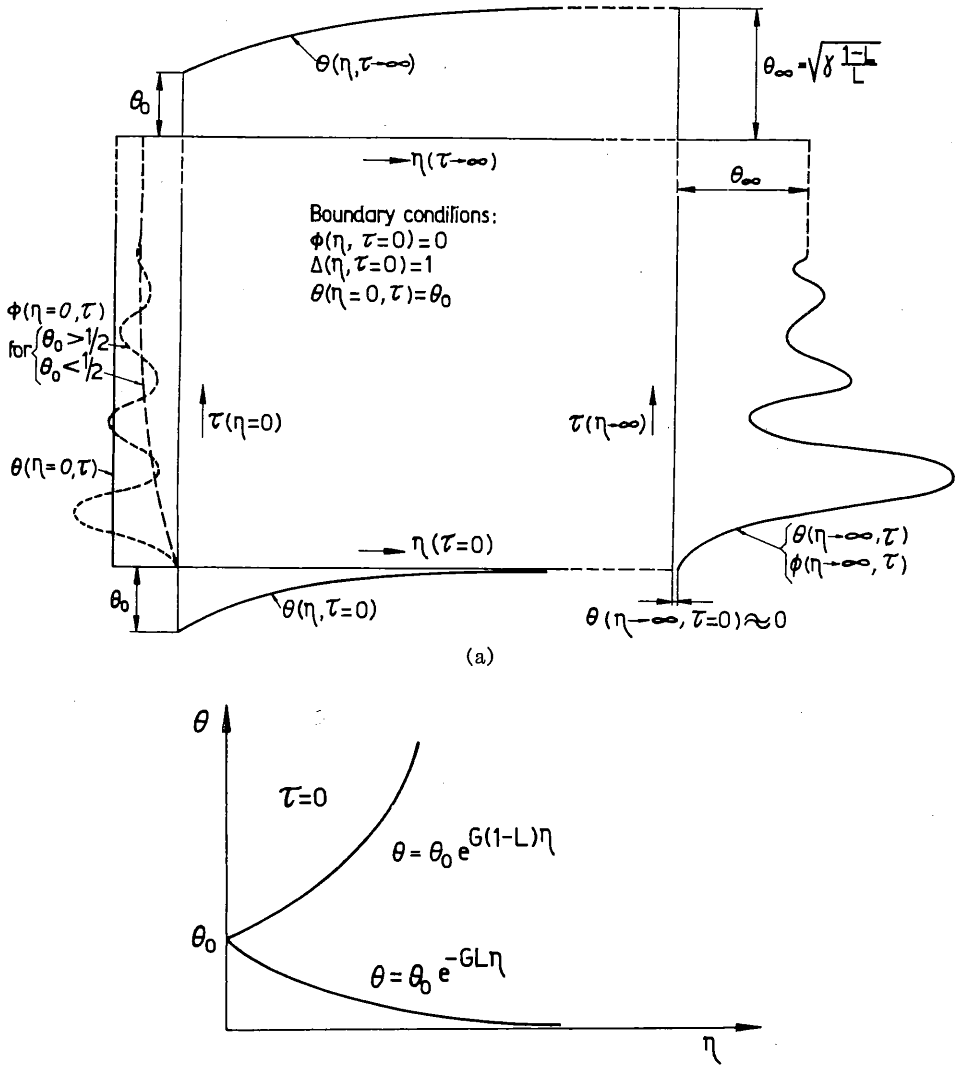
Instead of studying the general solutions, it is interesting to study the limit behavior on four lines (Fig. 1) of the $\eta - \tau$ plane

a) $\tau = 0, \qquad \eta \geq 0$

b) $\eta = 0, \qquad \tau \geq 0$

c) $\tau \to \infty, \qquad \eta \geq 0$

d) $\eta \to \infty, \qquad \tau \geq 0.$



(a)

(b)

Fig. 1. (a) Lines of the $\eta - \tau$ plane where the propagation problem of a step pulse is studied. (b) Comparison of the Θ vs. η diagrams at the initial time following (12) (exact solution) or (15) (approximation of Frantz and Nodvik [3]).

On some of these limit lines solutions already known from previous studies are found and here reduced to particular cases of a more general treatment. On the other limit lines new consequences are shown which are intimately connected to the detailed mechanism of the quantum amplifier and which could neither be explored by an elementary energy balance, as done in the rate-equation approach, nor by a more sophisticated atomic model which yet neglected the transport properties of the field.

Case a) corresponds to the case of illumination when the induced dipole strength is still zero and the field can interact only with the loss mechanism.

Case b) corresponds to studying the interaction at the initial section of the amplifier, where reaction on the field can be neglected, and the problem is therefore reduced to the well-known one of two-level pulse inversion [2].

Case c) corresponds to the asymptotic behavior when the induced dipole moment is in equilibrium with the in verted population as is assumed in the rate-equation approach.

Case d) generalizes the S.S.P. taking into account the finite value of T_1.

Case a) $\tau = 0$

Before the onset of an induced dipole moment, the loss contribution only is effective in the third part of (11) and, therefore, the solution is

$$\Theta(\eta, 0) = \Theta_0 e^{-GL\eta}. \tag{12}$$

That is, the initial part of the pulse is not amplified but decays to zero, whereas in a rate-equation approach [3], even when losses have been introduced, the material still behaves as an amplifier at the initial time, as shown in Fig. 1(b). The result might seem paradoxical at first sight, especially if one takes into account the assumption of a step-like incoming field. Actually, this is the behavior of a sharp front, whose leading edge has a rise time short in comparison to the build-up time of the induced dipoles, whereas the loss mechanism is so fast that it reacts almost instantly with the incoming field. Physically these losses may be attributed to scattering centers which change the optical properties of the medium with their resonances far away from ω_0, so that the tail of their dispersion curve is almost flat around ω_0.

Case b) $\eta = 0$

In the first section of the amplifier the applied field is unperturbed and therefore (16) reduces to

$$\begin{aligned} \frac{\partial \Phi}{\partial \tau} &= \Theta_0 \Delta - \Phi \\ \frac{\partial \Delta}{\partial \tau} &= -\Theta_0 \Phi - \gamma[\Delta - 1]. \end{aligned} \tag{13}$$

This is the well-known case of illumination of a two-level system by a strong field, and does not require further comment. Solutions for Φ and Δ are of the type exp $(\alpha\tau)$, where

$$\alpha = \frac{-(1 + \gamma) \pm i\sqrt{4\Theta_0^2 - (1 - \gamma)^2}}{2},$$

that is, Φ and Δ decay toward the asymptotic values

$$\begin{aligned} \Delta_\infty &= \frac{\gamma}{\gamma + \Theta_0^2} \\ \Phi_\infty &= \Theta_0 \Delta_\infty \end{aligned} \tag{14}$$

with an oscillating behavior, provided that

$$\Theta_0 > (1 - \gamma)/2.$$

This condition is always fulfilled for any input when γ is near to unity (which is a good approximation for the Ne laser). Both cases of oscillatory and monotonical polarization Φ are plotted qualitatively in Fig. 1(a).

Case c) $\tau \to \infty$

For local times much longer than T_2, but still short in comparison to T_1, that is, for $\tau \gg 1$ but $\tau \ll 1/\gamma$, the equilibrium value $\Phi = \Delta\Theta$ is reached while Δ has not yet reached a stationary condition. In this case equations (11) reduce to

$$\begin{aligned} \frac{\partial \Delta}{\partial \tau} &= -\rho\Delta - \gamma(\Delta - 1) \\ \frac{\partial \rho}{\partial \eta} &= 2G\rho(\Delta - L), \end{aligned} \tag{15}$$

where $\rho = \Theta^2$ is proportional to the photon density. These are the transport equations in the rate-equation approach [3] and this derivation from more general principles shows their limits of validity. For $\tau \to \infty$, Δ goes to the value

$$\Delta(\eta) = \gamma/[\gamma + \rho(\eta)] \tag{16}$$

which generalizes (14) for $\eta > 0$, and ρ is given by

$$\frac{\rho_0^L}{|\rho_0 - \rho_\infty|} \frac{|\rho(\eta) - \rho_\infty|}{\rho^L(\eta)} = \exp\left[-2GL(1 - L)\eta\right] \tag{17}$$

where

$$\rho_\infty = \gamma \frac{1 - L}{L} \tag{18}$$

is the value of ρ for $\eta \to \infty$. From (17) taking the square root of $\rho(\eta)$, one obtains the plot of Θ vs. η which is reported at the top of Fig. 1(a) for given values of G and L. Using (17) one might as well obtain the plot of Δ vs. η.

Case d) $\eta \to \infty$

From a physical viewpoint, one may expect that, due to the competition between gain and losses, a stationary condition is reached eventually in which the pulse propagates without changing shape and size (S.S.P.). Let us look for a solution of the transport problem corresponding to a pulse moving with velocity c without changes. This corresponds to put the Lagrangian derivative of Θ equal to zero, that is,

$$\frac{\partial \Theta}{\partial \eta} = 0 \tag{19}$$

which is equivalent to

$$\Phi = L\Theta. \tag{19a}$$

We will show now that condition (19) can be satisfied for $\eta \to \infty$. The third part of (11) can easily be integrated, and gives

$$\Theta(\eta, \tau) = \Theta(0, \tau)e^{-GL\eta} + Ge^{-GL\eta}\int_0^\eta e^{GL\eta'}\Phi(\eta', \tau)\, d\eta',$$

the limit of which for $\eta \to \infty$ is for any applied field $\Theta(0, \tau)$

$$\lim_{\eta\to\infty} \Theta(\eta, \tau) = \lim_{\eta\to\infty} \frac{G\int_0^\eta e^{GL\eta'}\Phi\, d\eta'}{e^{GL\eta}} = \frac{1}{L}\lim_{\eta\to\infty} \Phi(\eta, \tau)$$

which is the equality (19a). In this way, the existence of the S.S.P. has been proved for a general input shape $\Theta(0, \tau)$ demonstrating that the S.S.P. has no memory of the applied field at the input section. Introducing the S.S.P. condition into the general equations reduces them to the following equations in $\rho = \Theta^2$ and Δ

$$\begin{aligned} \frac{d\rho}{d\tau} &= 2\rho\left(\frac{\Delta}{L} - 1\right) \\ \frac{d\Delta}{d\tau} &= -L\rho - \gamma(\Delta - 1). \end{aligned} \tag{20}$$

The S.S.P. has already been introduced in Wittke and Warter [5] by numerical computations, and for the limit case $\gamma = 0$. Here it appears rigorously justified, and furthermore can be dealt with in the general case $\gamma \neq 0$. Since it has many physical applications it requires a more detailed treatment as will be presented in Section IV.

IV. The Steady-State Pulse

A. *General case $\gamma \neq 0$*

The S.S.P. is given by (20). A general solution cannot be worked out analytically, due to nonlinearities. The asymptotic solution for $\tau \to \infty$ is given by

$$\begin{aligned} \Delta_\infty &= L \\ \rho_\infty &= \gamma\frac{1 - L}{L} \end{aligned} \tag{21}$$

By a linear expansion around this singular point, one can study the behavior of the pulse for very long times. The latter is damped oscillator, with a damping time $2T_1$ and a pulsation

$$\omega = \sqrt{\frac{2}{T_1T_2}\frac{1 - L}{L} - \frac{1}{4T_1^2}}$$

which is real until $\gamma < 8(1 - L)/L$, that is, when

$$L < \frac{1}{1 + \gamma/8}. \tag{22}$$

For instance, for $\gamma = 0$ and $\gamma = 1$ this implies that L is less than 1 or $\frac{8}{9}$, respectively. In a real amplifier, with relatively small losses, condition (22) is always fulfilled and therefore the singularity is a focus in the phase plane $\Delta - \rho$. A study of the characteristics of the differential system (20) has been carried out in the plane $\Delta - \rho$ with an Olivetti-Elea 6001 computer. The system has two singular points, a saddle point corresponding to the equilibrium with the pump

$$P_0 \equiv (\rho = 0, \Delta = 1) \tag{23}$$

and the focus P_∞ attained for $\tau \to \infty$ and already considered in (21). The straight line which connects the two singularities is a locus of extremal points (maxima and minima of Δ vs. time), the horizontal line which crosses P_∞ is the locus where the characteristics have vertical tangent (maxima and minima of ρ vs. time). In Fig. 2 two examples are plotted for different values of L and incoming radiation strength, and a qualitative plot of Θ vs. τ is reported on the right side of Fig. 1(a). An interesting problem arises in connection with a laser amplifier with $\gamma = 1$, namely to discriminate between cases where Δ does not change sign and cases where Δ crosses the zero line during the oscillations. In the first case an inverted population Δ acts as an amplifying medium, whereas in the second case in a suitable time interval a coherent two-level inversion might occur, as in the paramagnetic case. As one easily realizes from Fig. 2, the value of L separating the two cases is that having a particular characteristic which in its first loop, starting from the line of the initial conditions $\Delta = 1$, crosses the locus of extrema at the intersection with the ρ axis. This occurs for a value L between 0.3 and 0.4. Thus, for $L \leq 0.3$ a two-level inversion would be possible. The case $\gamma = 1$ has been considered here since it corresponds to a set of experimental gaseous lasers, namely those for which $T_2 \simeq T_1$, that is, those which work at such low pressure that collision exchange is negligible with respect to the spontaneous decay occuring to the usual He-Ne laser [6].

B. *Case $\gamma = T_2/T_1 = 0$*

Equations (20) have an analytical solution in the particular case $\gamma = 0$, that is, when $T_1 \gg T_2$, so that one can neglect the relaxation of the diagonal elements of the density matrix. This analytical solution is fitted by the numerical computations of Wittke and Warter [5]. When $\gamma = 0$, (20) yields the relation:

$$\frac{d^2\Delta}{d\tau^2} + 2\frac{d\Delta}{d\tau}\left(1 - \frac{\Delta}{L}\right) = 0 \tag{24}$$

which can easily be solved for the initial conditions $\Delta(0) = 1$, $\rho(0) = \rho_0$ (where ρ_0 is very small). Denoting by Δ_1 and Δ_2 the roots of the equation $d\Delta/d\tau = 0$, that is,

$$\Delta_{1,2} = L \pm \sqrt{(1 - L)^2 + \rho_0L}$$

which represents the points where $\rho = 0$, and using the notation

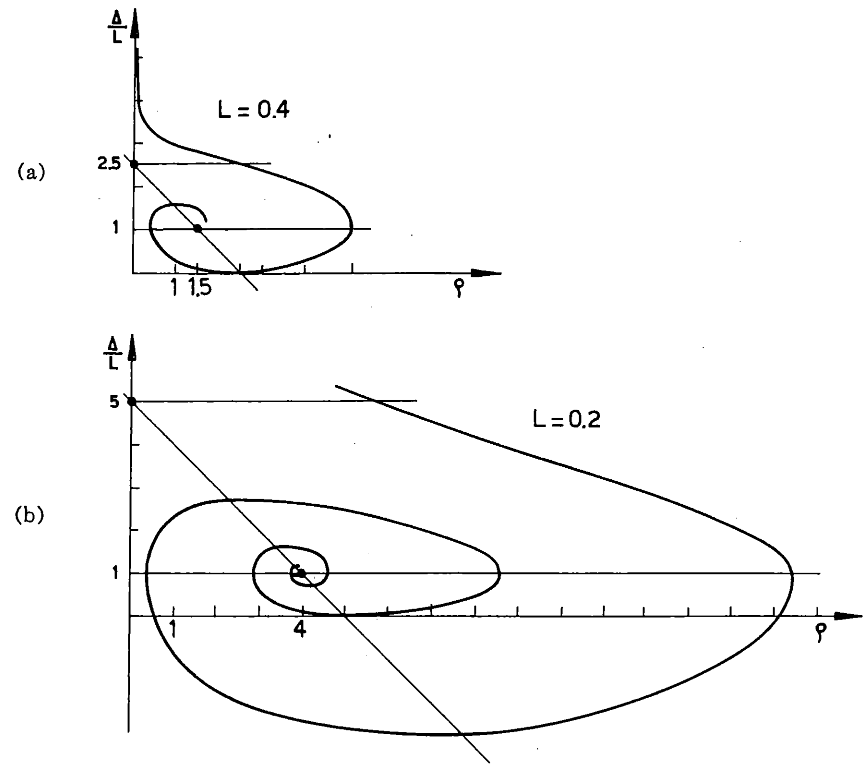


Fig. 2. Plot of the characteristics $\Delta - \rho$ for two different values of L. The line $\Delta/L = 1/L$ is the locus of the initial conditions for $\tau = 0$. A physical characteristic must start with an initial value $\rho_0 \ll 1$. (a) $L = 0.4$. In this case, the characteristic with $\rho_0 \simeq 2.5$ is tangent to the ρ axis in its first loop, therefore a physical characteristic can not cross the ρ axis, and the population difference never changes sign. (b) $L = 0.2$. In this case, the characteristic with $\rho_0 \simeq 5$ is tangent in its second loop and obliges the characteristics with lower ρ_0 to cross the line $\Delta = 0$. Two-level inversion can therefore be obtained on some interval of time.

$$K = \frac{1 - \Delta_2}{\Delta_1 - 1}$$

$$\alpha = \frac{\Delta_1 - \Delta_2}{L},$$

the solutions for Δ and ρ are

$$\Delta(\tau) = \frac{\Delta_1 K e^{-\alpha\tau} + \Delta_2}{1 + K e^{-\alpha\tau}} \tag{25}$$

$$\rho(\tau) = 4\rho_M \frac{K e^{-\alpha\tau}}{(1 + K e^{-\alpha\tau})^2} \tag{26}$$

where

$$\rho_M = \alpha^2 = \rho_0 + \left(\frac{1 - L}{L}\right)^2$$

is the peak value of the photon density. Since

$$\rho_0 \ll (1 - L)^2/L^2,$$

that is, the initial photon density can be neglected compared to the maximum of the S.S.P., one can approximate

$$\Delta_1 \simeq 1, \qquad \Delta_2 \simeq 2L - 1.$$

We notice by direct investigation of system (20) with $\gamma = 0$, that:

a) Δ decreases monotonically with time,
b) the photon density has a maximum for $\Delta = L$,
c) the induced dipole moment $\Phi = L\Theta$ represents the square root of ρ.

Furthermore, relation (26) shows that:

a) The photon pulse above ρ_0 is symmetrical around its maximum, and returns to zero for $\Delta = 2L - 1$. Its half-width is given by

$$\tau_{1/2} = \frac{L}{1 - L} \ln (3 + \sqrt{8}). \tag{27}$$

b) The area of the pulse above the initial value is given by

$$A = 2\frac{1 - L}{L} = 2\sqrt{\rho_M - \rho_0}. \tag{28}$$

A qualitative plot of ρ and Δ vs. τ is shown in Fig. 3 for a value of L less than 0.5.

C. *Practical considerations on the S.S.P.*

The time τ_m for which ρ is maximum is given, for $\rho_0 \ll 1$,

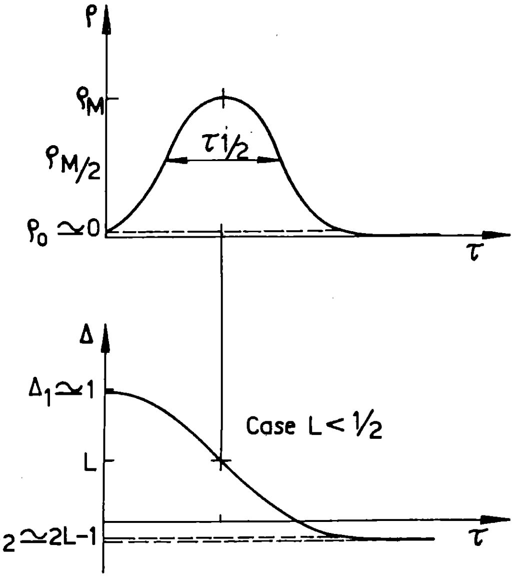


Fig. 3. Diagram of Δ and ρ vs. τ for a value $L < 1/2$ (final value of Δ below zero), when $T_2/T_1 = 0$.

by

$$\tau_m = \frac{L}{2(1-L)} \ln \frac{4(1-L)}{\rho_0 L} \tag{29}$$

and becomes very long when ρ_0 goes toward zero. Actually $\rho_0 \to 0$ is a limit situation, and in practical cases the S.S.P. condition is approximately satisfied when the incoming pulse has been reduced by losses to a value $\rho_0 \ll 1$. For instance, if one takes $\rho_0 = 10^{-8}$, which corresponds, for a material like ruby, with an oscillator strength $f = 10^{-6}$ and $T_2 \simeq 10^{-10}$ sec [7], to a residual power flow of about 10 W/cm^2, we have the following values of τ_m for two different values of L:

$$L = 0,55, \qquad \tau_m \simeq 10$$

$$L = 0,99, \qquad \tau_m \simeq 140.$$

One sees that the maxima of the S.S.P. can occur for local times much longer than T_2, and therefore can be observed with usual techniques even for very short values of T_2, once high loss values are provided.

One may ask how long a maser rod should be in order to achieve the S.S.P. condition at the end. From a practical viewpoint, the condition $\partial\rho/\partial\eta = 0$ is almost satisfied for a length η much longer than $\frac{1}{2}$GL. For instance for a maser medium with scattering losses of 10 percent per cm, one should use a length large compared to 10 cm.

V. Short-Time Solution in a Lossless Amplifier

When $L = 0$, there is no S.S.P. as one can see from (19a). For times long compared to T_2, this case has already been studied using the rate-equation approach [3]. It is now interesting to study the behavior of the lossless amplifier for $\tau \ll 1$. In this case equations (11) reduce to the following:

$$\frac{\partial\Phi}{\partial\tau} = \Theta\Delta$$

$$\frac{\partial\Delta}{\partial\tau} = -\Theta\Phi \tag{30}$$

$$\frac{\partial\Theta}{\partial\eta} = G\Phi$$

with the boundary conditions (10). Since the first two equations lead to the following relation

$$\frac{\partial}{\partial\tau}(\Phi^2 + \Delta^2) = 0$$

a transformation of variables can be made[1]

$$\Phi = r \sin \xi$$
$$\Delta = r \cos \xi \tag{31}$$

with the initial conditions

$$r(\eta, 0) = 1$$

$$\xi(\eta, 0) = 0.$$

Equations (30) become therefore

$$\frac{\partial r}{\partial\tau} = 0$$

$$\frac{\partial\xi}{\partial\tau} = \Theta \tag{32}$$

$$\frac{\partial\Theta}{\partial\eta} = Gr \sin \xi.$$

[1] The mathematical approach of this section has been proposed and worked out by Dr. C. Cercignagni and Dr. A. Ghirardi.

The first yields $r = 1$, the second and third yield

$$\frac{\partial^2 \xi}{\partial \tau \, \partial \eta} = G \sin \xi$$

which is equivalent to the integral equation

$$\xi(\eta, \tau) = \Theta_0 \tau + G \int_0^\tau d\tau' \int_0^\eta d\eta' \sin \xi(\eta', \tau') \qquad (33)$$

where the last term takes into account the boundary condition

$$\left(\frac{\partial \xi}{\partial \tau}\right)_{\eta=0} = \Theta_0.$$

Equation (33) can be solved in successive orders of approximation starting from $\eta = 0$. The solution at the 0th order is

$$\xi_0(\eta, \tau) = \Theta_0 \tau.$$

The solution at any order is given by the following recurrence formula

$$\xi_m = \xi_{m-1} + G \int_0^\tau d\tau' \int_0^\eta d\eta' \sin \xi_{m-1}(\eta', \tau'). \qquad (34)$$

The existence and uniqueness of a limit for $m \to \infty$ of this formula can be proved by an extension of a well-known theorem on the Volterra nonlinear integral equation of the second kind [8]. The solution at the first order is

$$\xi_1(\eta, \tau) = \Theta_0 \tau + \frac{G\eta}{\Theta_0}(1 - \cos \Theta_0 \tau) \qquad (35)$$

from which one finds

$$\Theta_1 = \Theta_0 + G\eta \sin \Theta_0 \qquad (36)$$

Equation (36) shows that, at the first order, the field is affected by a term oscillating in the local time τ with a frequency $\Theta_0 T_2 = E_0 \mu/\hbar$ and with an amplitude increasing linearly along the amplifier. From (35) one finds the following relation for the induced dipole

$$\Phi_1 = \sin\left[\Theta_0 \tau + \frac{G\eta}{\Theta_0}(1 - \cos \Theta_0 \tau)\right] \qquad (37)$$

where the 0th order contribution is the well-known term given from the usual theory for a two-level system [9], and the second term takes into account the propagation effects when the field increases with the law (36). For very high fields the second term vanishes, and Φ reduces to the relation used in the two-level pulse inversion technique and usually calculated in the approximation of an unperturbed pumping field.

VI. Asymptotic Evaluations for Field Out of Resonance

When the field is not at resonance with the atomic transition, that is, $\delta_0 = \omega - \omega_0 \neq 0$ one has to work on the general set of (5). However the problem is quite simplified if one looks only for the asymptotic behavior at large η's, which has already appeared as a particularly interesting situation.

We show that also in the off-resonance case there is an S.S.P., that is, the asymptotic pulse propagates without changing shape or size, and its Lagrangian derivatives $\partial\Theta/\partial\eta$ and $\partial\varphi/\partial\eta$ are zero.

That $\partial\Theta/\partial\eta$ is zero is easily proved as shown in Section III. That $\partial\varphi/\partial\eta$ is zero is proved by direct substitution; namely from the fourth and second parts of (5); $\partial\varphi/\partial\eta = 0$ implies that $\Psi = 0$ identically and therefore

$$\frac{\partial \varphi}{\partial \tau} = -\delta_0. \qquad (38)$$

Now the solution at $\tau = 0$ of (5) for φ is $\varphi(\eta, \tau = 0) = 0$ and the integral of (38) with this initial value is therefore

$$\varphi = -\delta_0 \tau \qquad (39)$$

which is also consistent with the other parts of (20). Therefore, for the uniqueness of the solution, the initial assumption $\partial\varphi/\partial\eta = 0$ is correct and the asymptotic solution is an S.S.P. solution with φ given by (39).

The equations of the S.S.P. are formally identical to (20) but with the following fundamental difference. Here the cosine argument of relation (1) is given by

$$\omega\tau - \delta_0 \tau = \omega_0 \tau, \qquad (40)$$

that is, the central frequency of the S.S.P. field coincides with the center frequency of the atomic line. At first sight this result might not seem consistent with the principle of stimulated emission, which would suggest that the amplified contribution should belong to the same frequency of the incoming wave. A physical explanation of this phenomenon must take into account the finite spectral width of the incoming field. The spectrum of the em pulse (1) even if not centered at the atomic transition frequency, has a tail at ω_0 because of amplitude and phase modulation. Now, even if initially the spectral component Θ_ω of the field at ω is much larger than the component Θ_{ω_0} at ω_0, then, in the course of the propagation there will be a continuous decrease of the ratio $\Theta_\omega/\Theta_{\omega_0}$ up to the asymptotic value $\Theta_\omega/\Theta_{\omega_0} = 0$ since the gain is larger at resonance, whereas the relative losses do not depend on the frequency. The S.S.P. situation is therefore an asymptotic one of balance between gain and losses, where memory of the input field has been completely lost, and the propagating field depends only on the features of the medium. This situation is analogous to that of an oscillator, with the difference here that the frequency is not determined by the resonant properties of an em cavity but only by the material. A scheme of this kind would be a better frequency standard than an oscillator, where the frequency of the cavity is subject to external perturbations. The spectral width of the S.S.P. is defined only by the spectrum of the envelope, since the phase goes linearly with the time. It follows that the spectral width depends on the loss factor and for high losses decreases as $(1 - L)/L$, as shown by (27). Therefore a long amplifier suitable for yielding a standard frequency pulse should have high losses.

These considerations are similar, although inferred in a somewhat different way, to some ideas developed recently on the problem of the frequency standard [10], [11].

VII. Conclusions

The theory developed here describes in a general way the interaction between a maser medium and a field not necessarily at resonance with the maser transition. Particular emphasis has been given to solutions of four regions, corresponding, respectively, to a) no induced polarization, b) unperturbed field, c) long times (rate-equation approach), and d) large lengths (S.S.P. condition). The last case has appeared particularly interesting and when a usual laser material is doped with quite a large density of scattering centers it appears already detectable for small lengths. The results presented here are quite different from previous treatments. The most interesting features of the S.S.P. are first, that it propagates at the velocity of the light in the medium, and second, that when the field is out of resonance, the S.S.P. field frequency is moved away from the input frequency and coincides with the atomic transition frequency. This last fact could be used to design a frequency standard.

Appendix I

With reference to the assumption of Section I and II, the interaction coupling electric field and dipole moment μ of each atom at a given point in space gives rise to an equation of motion for the density matrix ρ. Using the interaction representation, the motion can be represented in a 1, 2, 3 frame rotating at an angular velocity ω_0 equal to the frequency difference of the two atomic levels by the motion of a vector $\mathbf{R}$ whose components are the following linear combinations of the ρ components (see Vuylsteke [9] for notations):

$$\begin{aligned} R_1 &= \rho_{12}^* + \rho_{21}^* \\ R_2 &= i(\rho_{12}^* - \rho_{21}^*) \\ R_3 &= \rho_{11}^* - \rho_{22}^* = \rho_{11} - \rho_{22}. \end{aligned} \tag{41}$$

The equation of motion for $\mathbf{R}$ is

$$\frac{\partial \mathbf{R}}{\partial t} = \frac{1}{T_1}\left(\frac{D_e}{N}\right)\hat{e}_3 - \mathcal{B}\cdot\mathbf{R} + \mathbf{\Omega}\times\mathbf{R}. \tag{42}$$

In this equation, $\hat{e}_i$ is the unit vector of the i axis (i = 1, 2, 3), D_e is the population difference at equilibrium with the pump, N is the active atom density, the matrix product $\mathcal{B}\cdot\mathbf{R}$ accounts for the relaxation processes,

$$\mathcal{B}\cdot\mathbf{R} = \frac{1}{T_2}R_1\hat{e}_1 + \frac{1}{T_2}R_2\hat{e}_2 + \frac{1}{T_1}R_3\hat{e}_3, \tag{43}$$

and the $\mathbf{\Omega}$ vector stands for the interaction

$$\begin{aligned} \Omega_1 &= \Lambda\cos[(\omega - \omega_0)t - kx + \varphi(x, t)] \\ \Omega_2 &= -\Lambda\sin[(\omega - \omega_0)t - kx + \varphi(x, t)] \end{aligned} \tag{44}$$

where Λ is the amplitude of the interaction in angular frequency units:

$$\Lambda = \frac{E_0\mu}{\hbar}. \tag{45}$$

Until now we have followed the treatment of Vuylsteke [9], with the following main changes. First, our formulas depend on both time and space, and therefore the derivatives are partial and x denotes the position of the center of a given atom. Second, we use a real notation instead of a complex one and therefore relations (5.24) of Vuylsteke [9], lead to our relations (44) by the substitution here shown,

$$H_{12}^* = H_{12}e^{-i\omega_0 t}, \qquad H_{21}^* = H_{21}e^{i\omega_0 t}$$

and

$$H_{21} = H_{12} = \mu E_0 \cos[\omega t - kx + \varphi(x, t)]$$

having assumed μ as real. Now we make a change of reference in order to write equation (42) in a more suitable form. The $\mathbf{\Omega}$ vector lies on the 1–2 plane and rotates at a velocity proportional to the off resonance

$$\delta = \omega - \omega_0 + \frac{\partial\varphi}{\partial t}, \tag{46}$$

i.e., its unit vector $\hat{n}$ rotates as

$$\frac{\partial\hat{n}}{\partial t} = -\delta\hat{e}_3\times\hat{n}.$$

Equation (42) can be rewritten in a new frame whose 3 axis coincides with the previous one, whereas its 2 axis coincides with the $\mathbf{\Omega}$ direction. The $\mathbf{R}$ vector in this new reference still obeys an equation like (42) where now $\mathbf{\Omega}$ is given by

$$\begin{aligned} \Omega_1' &= 0 \\ \Omega_2' &= \Lambda \\ \Omega_3' &= \delta. \end{aligned}$$

The components of (42) become in this reference

$$\begin{aligned} \frac{\partial R_1'}{\partial t} &= -\delta R_2' + \Lambda R_3 - \frac{R_1'}{T_2} \\ \frac{\partial R_2'}{\partial t} &= \delta R_1' - \frac{R_2'}{T_2} \\ \frac{\partial R_3}{\partial t} &= -\Lambda R_1' - \frac{1}{T_1}(R_3' - D_e/N). \end{aligned} \tag{47}$$

The physical meaning of the three components is the following. R_3' is proportional to the inverted population difference $D = n_2 - n_1$, R_2' and R_1' are, respectively, proportional to the "in-phase" and "in-quadrature" components of the induced macroscopic polarization (3), that is,

$$\begin{aligned} C &= -\mu NR_2' \\ S &= \mu NR_1' \\ D &= -NR_3'. \end{aligned} \tag{48}$$

The relationships between the C and S components of the polarization and the R_2' and R_1' components of the R' vector are easily proved as follows. The volume polarization is the statistical average of the moment operator $\mathfrak{M}$ over the ensemble, that is,

$$P(x, t) = N\langle \mathfrak{M} \rangle = NTr\{\rho \mathfrak{M}\} = -N\mu(\rho_{12} + \rho_{21}). \quad (49)$$

since $\mathfrak{M}$ has only off-diagonal elements equal to $-\mu$. If we now write in the interaction representation and take into account (41) and the transformations formulas

$$\rho_{12} = \rho_{12}^{*} e^{i\omega_0 t},$$

$$\rho_{21} = \rho_{21}^{*} e^{-i\omega_0 t},$$

P can be written as

$$P(x, t) = -N\mu(R_1 \cos \omega_0 t + R_2 \sin \omega_0 t).$$

The second change of reference corresponds to the following transformation of the R components

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} -\sin \alpha & \cos \alpha \\ -\cos \alpha & -\sin \alpha \end{bmatrix} \begin{bmatrix} R_1' \\ R_2' \end{bmatrix}$$

where

$$\alpha = (\omega - \omega_0)t - kx + \varphi(x, t).$$

Making this substitution, we arrive at

$$P(x, t) = N\mu R_1' \sin [\omega t - kx + \varphi(x, t)] - N\mu R_2' \cos [\omega t - kx + \varphi(x, t)] \quad (50)$$

which proves (48).

Appendix II

Derivation of the Transport Equations for the EM Field

We derive two transport equations for the amplitude and phase of the electric field (1) under assumptions (2) and with the polarization by (3) as source term. Let us take a volume distribution of losses, represented by a constant conductivity σ. Maxwell equations written in mks units, reduce in our one-dimensional case to (see, for instance, equation (3) of Lamb, [12])

$$\frac{\partial^2 E}{\partial x^2} = \mu_0 \sigma \frac{\partial E}{\partial t} + \mu_0 \epsilon \frac{\partial^2 E}{\partial t^2} + \mu_0 \frac{\partial^2 P}{\partial t^2}$$

where μ_0 and ϵ are the em parameters of the medium which includes the active two-level systems, and hence $c = (\epsilon\mu_0)^{-1/2}$ is the light velocity in the medium. From this equation, using (1) and (3) together with assumptions (2), and equating separately sine and cosine terms, we arrive at the following transport equations

$$\begin{cases} E_0\left(c \dfrac{\partial \varphi}{\partial x} + \dfrac{\partial \varphi}{\partial t}\right) = \dfrac{\omega}{2\epsilon} C \\ c \dfrac{\partial E_0}{\partial x} + \dfrac{\partial E_0}{\partial t} + \dfrac{\sigma}{2\epsilon} E_0 = -\dfrac{\omega}{2\epsilon} S, \end{cases} \quad (51)$$

provided a slowly varying assumption similar to (2) is made for P, so that

$$\frac{\partial^2 P}{\partial t^2} \simeq -\omega^2 P.$$

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